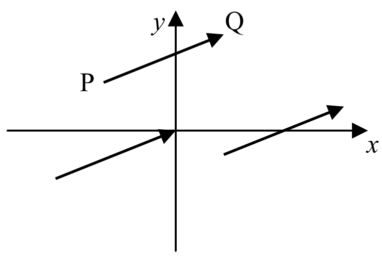
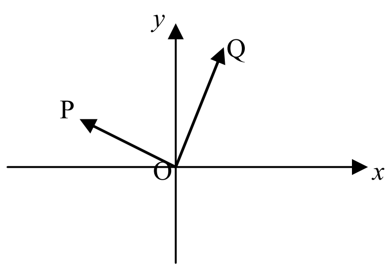
Vectors and vector calculus

Vectors in two and three dimensions

Free vectors: A vector is free if it is considered to be the same vector under translation. $\overrightarrow{PQ}$ is a free vector because it can be repositioned as shown in the following diagram.



A position vector is not a free vector because it cannot be translated. It always starts from a reference point O, the origin. $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ shown below are position vectors.



Most vectors can be added or subtracted if they represent the same quantity.

Exceptions:

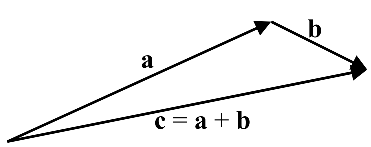
1. Addition of two position vectors is undefined.

2. A displacement vector can be added to a position vector to give a new position vector.

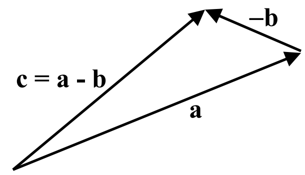
3. Subtraction of 2 position vectors is defined as displacement.

4. Force vectors acting on different objects cannot be added.

Addition of vectors is carried out by putting the head of one to the tail of the other. The order is irrelevant.



Subtraction is done by adding the second vector in reverse direction to the first vector.



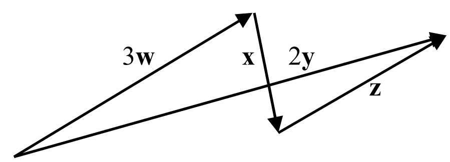
Linear dependence and independence

Vectors **x**, **y** and **z** are linearly dependent if there exist non-zero scalars p, q and r such that

$$p\mathbf{x} + q\mathbf{y} + r\mathbf{z} = \mathbf{0}.$$

If $p\mathbf{x} + q\mathbf{y} + r\mathbf{z} = \mathbf{0}$ only when $p = q = r = 0$, then **x**, **y** and **z** are linearly independent.

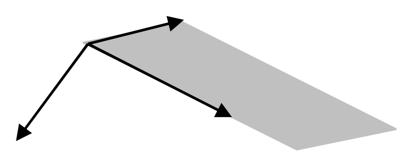
Example 1



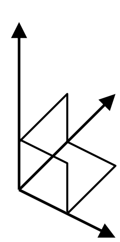
Since $3\mathbf{w} + \mathbf{x} + \mathbf{z} = 2\mathbf{y}$, i.e. $3\mathbf{w} + \mathbf{x} - 2\mathbf{y} + \mathbf{z} = 0$,

Therefore, **w**, **x**, **y** and **z** are linearly dependent.

Example 2 Three non-coplanar vectors are linearly independent.



A typical example is the case where the three vectors are perpendicular to each other.



Example 3 Given **a** = 3 units north and **b** = 4 units east, find a vector **c** which is linearly dependent on **a** and **b**. Find another vector **d** which is also linearly dependent on **a** and **b**.

A possible vector **c** is **c** = **a** + **b** because **c** – **a** – **b** = **0**; a possible vector **d** is **d** = **a** – 2**b** because **d** – **a** – 2**b** = **0**.

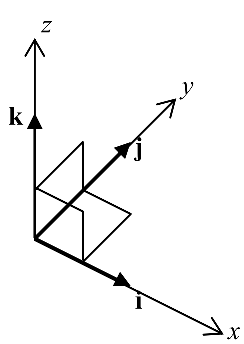
Example 4 Given that vectors **p**, **q** and **r** are linearly independent, and $a^2(\mathbf{p}+\mathbf{q}-\mathbf{r}) = (a+6)\mathbf{p} + (2a+3)\mathbf{q} - 9\mathbf{r}$, find the value(s) of a.

$a^2(\mathbf{p}+\mathbf{q}-\mathbf{r}) = (a+6)\mathbf{p} + (2a+3)\mathbf{q} - 9\mathbf{r}$,
$\therefore (a^2-a-6)\,\mathbf{p} + (a^2-2a+3)\,\mathbf{q} - (a^2-9)\,\mathbf{r} = \mathbf{0}$.
Since **p**, **q** and **r** are linearly independent,
$\therefore a^2-a-6 = a^2-2a+3 = -(a^2-9) = 0$,
i.e. $(a+2)(a-3) = (a+1)(a-3) = -(a+3)(a-3) = 0$.
$a = 3$ is the only value that satisfies all three equations.

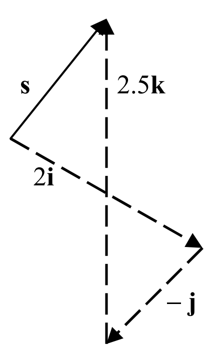
Unit vectors

Any vector with a magnitude of 1 is called a unit vector. A unit vector in the direction of vector **s** is labelled as **ŝ**. It is found by dividing vector **s** by its magnitude |**s**|, i.e. **ŝ = s**/|**s**|.

There are three unit vectors, **i**, **j** and **k** that are particularly useful in vector analysis. **i**, **j** and **k** are in the direction of x, y and z axes respectively. They are perpendicular to each other and therefore, linearly independent, i.e. if $p\mathbf{i} + q\mathbf{j} + r\mathbf{k} = \mathbf{0}$, then $p = q = r = 0$.



Any vector can be written in terms of **i**, **j** and **k**, e.g. **s** = 2**i** – **j** + 2.5**k**.



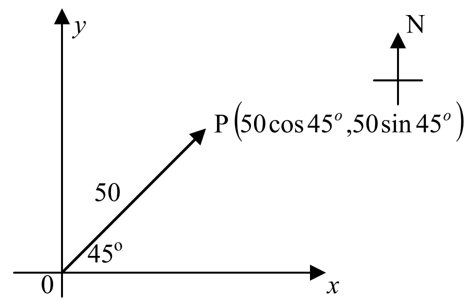
s, **i**, **j** and **k** are linearly dependent.

Resolution of a vector into rectangular components

A vector is resolved into rectangular components when it is expressed in terms of **i**, **j** and **k**, e.g. $\mathbf{r} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$.

The magnitude of **r** is $|\mathbf{r}| = r = \sqrt{l^2+m^2+n^2}$.

Example 1 An aeroplane is 50 km NE of Melbourne Airport and it is flying at an altitude of 10 km. Resolve its position vector from Melbourne Airport into **i**, **j** and **k** components. They are respectively pointing east, north and up. What is the straight-line distance of the plane from the runway?



$\overrightarrow{OP} = 50\cos 45^o\,\mathbf{i} + 50\sin 45^o\,\mathbf{j} + 10\,\mathbf{k}$
$= 25\sqrt{2}\,\mathbf{i} + 25\sqrt{2}\,\mathbf{j} + 10\,\mathbf{k}$.
Distance from the runway $= \left|\overrightarrow{OP}\right| = \sqrt{50^2+10^2} = 10\sqrt{26}$ km

Example 2 Find the magnitude of **s** = 2**i** – **j** + 2.5**k**, and a unit vector in the direction of **s**.

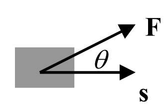
$|\mathbf{s}| = s = \sqrt{2^2+(-1)^2+2.5^2} = \dfrac{3\sqrt{5}}{2}$.
Unit vector in the direction of **s**:
$\hat{\mathbf{s}} = \mathbf{s}/|\mathbf{s}| = (2\mathbf{i} - \mathbf{j} + 2.5\mathbf{k})/\dfrac{3\sqrt{5}}{2} = \dfrac{2}{3\sqrt{5}}(2\mathbf{i} - \mathbf{j} + 2.5\mathbf{k})$.

Scalar (dot) product of two vectors

Scalar (dot) product of two vectors is defined to meet the requirements of many physical situations,

e.g. in physics work done W joules by a force **F** newtons in displacing an object **s** metres is

$W = Fs\cos\theta$



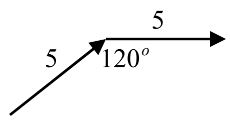
If the scalar product of **F** and **s** is defined as $\mathbf{F.s} = Fs\cos\theta$, then W = **F.s**.

The scalar product of two vectors gives a scalar.

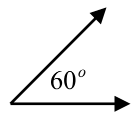
Note: (1) In a scalar product, θ is the angle between two vectors that are placed tail to tail.

(2) $\mathbf{r.r} = r^2$, $\therefore r = \sqrt{\mathbf{r.r}}$

Example 1 Find the scalar product of the two vectors.



Place the two vectors tail to tail, $\theta = 60^o$.



Scalar product $= 5 \times 5 \cos 60^o = 12.5$.

Parallel and perpendicular vectors

Two vectors, **p** and **q**, are parallel if one equals the other multiplied by a constant, **p** = α**q**.

Two parallel vectors are linearly dependent.

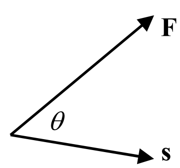
The angle between two parallel vectors is either zero or 180^o.
$\therefore$ **F.s** = Fs or –Fs.

Two vectors are perpendicular ($\theta = 90^o$) if **F.s** = 0.
Conversely, **F.s** = 0 if two vectors are perpendicular.

Two perpendicular vectors are linearly independent.

Note that **i.i** = **j.j** = **k.k** = 1 and **i.j** = **j.k** = **k.i** = 0.

Scalar product of vectors in i, j, k components



If **F** and **s** are in terms of **i**, **j** and **k**, i.e. **F** = a**i** + b**j** + c**k** and **s** = x**i** + y**j** + z**k**, then

F.s = (a**i** + b**j** + c**k**).(x**i** + y**j** + z**k**) = $ax + by + cz$.

Since **F.s =** $Fs\cos\theta$ **,** $\therefore \cos\theta = \dfrac{ax+by+cz}{Fs}$.

Hence $\cos\theta = \dfrac{ax+by+cz}{\sqrt{a^2+b^2+c^2}\sqrt{x^2+y^2+z^2}}$.

Example 1 Show that **p** = 3**i** + 2**j** + 4**k** is parallel to **q** = $\frac{1}{4}$**i** + $\frac{1}{6}$**j** + $\frac{1}{3}$**k**.

q = $\frac{1}{4}$**i** + $\frac{1}{6}$**j** + $\frac{1}{3}$**k** = $\frac{3}{12}$**i** + $\frac{2}{12}$**j** + $\frac{4}{12}$**k**
= $\frac{1}{12}$(3**i** + 2**j** + 4**k**) = $\frac{1}{12}$**p**. $\therefore$ **p** and **q** are parallel.

Example 2 Show that 3**i** + 2**j** + 4**k** is perpendicular to 2**i** + **j** – 2**k**.

(3**i** + 2**j** + 4**k**).(2**i** + **j** – 2**k**) = 6 + 2 – 8 = 0.
$\therefore$the two vectors are parallel.

Example 3 Find c such that 2**i** + 2**j** – **k** is perpendicular to **i** + c**j** + 6**k**.

(2**i** + 2**j** – **k**).(**i** + c**j** + 6**k**) = 0, $\therefore 2 + 2c - 6 = 0,\ c = 2$.

Example 4 Find a vector perpendicular to **g** = **i** – 2**j** + 3**k**.

Let **r** = l**i** + m**j** + n**k** be a vector perpendicular to **g**.
r.g = 0. $\therefore l - 2m + 3n = 0$.

Let $m = 0$ and $n = 1$, $\therefore l = -3$.
Hence **r** = –3**i** + **k** is a vector (out of an infinite number of them) perpendicular to **g**.

Example 5 Find a unit vector perpendicular to **p** = 2**i** – 3**j** and (a) on the same plane, (b) not on the same plane.

Let **u** = l**i** + m**j** + n**k** be a unit vector $\perp$ to **p** = 2**i** – 3**j**,
$\therefore \sqrt{l^2+m^2+n^2} = 1,\ l^2+m^2+n^2 = 1$ and $2l - 3m = 0$.

(a) If **u** is on the same plane as **p**, then the **k** component of **u** is zero, i.e. $n = 0$. Hence $l^2 + m^2 = 1$ ……(1)
and $2l - 3m = 0$ ……(2)

Solve the simultaneous equations for l and m:
From (2), $m = \frac{2}{3}l$ …...(3). Substitute (3) in (1),

$l^2 + \frac{4}{9}l^2 = 1,\ \frac{13}{9}l^2 = 1,\ \therefore l = \pm\frac{3}{\sqrt{13}} = \pm\frac{3\sqrt{13}}{13},\ \therefore m = \pm\frac{2\sqrt{13}}{13}.$

Hence **u** $= \frac{3\sqrt{13}}{13}$**i** $+ \frac{2\sqrt{13}}{13}$**j** or $-\frac{3\sqrt{13}}{13}$**i** $- \frac{2\sqrt{13}}{13}$**j**.

(b) **u** (or –**u**) is a unit vector $\perp$ to **p**, but not on the same plane as **p**. It contains **i**, **j** and **k** components.

$l^2 + m^2 + n^2 = 1$ ……(1) and, $2l - 3m = 0$ ……(2)
From (2), $m = \frac{2}{3}l$ …...(3). Substitute (3) in (1),

$l^2 + \frac{4}{9}l^2 + n^2 = 1,\ \therefore \frac{13}{9}l^2 + n^2 = 1,\ n = \pm\sqrt{1 - \frac{13}{9}l^2}.$

Choose $l = \frac{1}{\sqrt{13}} = \frac{\sqrt{13}}{13}$, then $m = \frac{2}{3\sqrt{13}} = \frac{2\sqrt{13}}{39}$ and

$n = \pm\sqrt{1 - \frac{13}{9}l^2} = \pm\sqrt{1 - \frac{1}{9}} = \pm\frac{2\sqrt{2}}{3}.$

Hence **u** $= \frac{\sqrt{13}}{13}$**i** $+ \frac{2\sqrt{13}}{39}$**j** $+ \frac{2\sqrt{2}}{3}$**k**, or

u $= \frac{\sqrt{13}}{13}$**i** $+ \frac{2\sqrt{13}}{39}$**j** $- \frac{2\sqrt{2}}{3}$**k**.

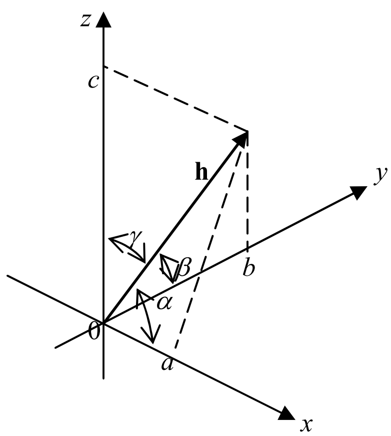
Direction cosines

Consider vector $\mathbf{h} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, and the unit vector in the direction of $\mathbf{h}$ is:

$$\hat{\mathbf{h}} = \frac{a}{\sqrt{a^2+b^2+c^2}}\,\mathbf{i} + \frac{b}{\sqrt{a^2+b^2+c^2}}\,\mathbf{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\,\mathbf{k},$$

where $\sqrt{a^2+b^2+c^2} = |\mathbf{h}|$.

Let the angles that the unit vector make with the x, y and z-axes be α, β, γ respectively.



$$\therefore \cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}},\ \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}$$

and $\cos\gamma = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$.

Hence $\hat{\mathbf{h}} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$.

$\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of $\mathbf{h}$.

Example 1 Vector $\mathbf{r}$ has a magnitude of 10, and it makes angles of 30^o, 45^o and 60^o respectively with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. Express $\mathbf{r}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$\mathbf{r} = 10\cos 30^o\,\mathbf{i} + 10\cos 45^o\,\mathbf{j} + 10\cos 60^o\,\mathbf{k}$

$\therefore \mathbf{r} = 5\sqrt{3}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j} + 5\,\mathbf{k}$.

Example 2 Find the magnitude and direction cosines of $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$.

Magnitude $= \sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$.

Unit vector $= \dfrac{3}{5\sqrt{2}}\mathbf{i} - \dfrac{4}{5\sqrt{2}}\mathbf{j} + \dfrac{5}{5\sqrt{2}}\mathbf{k}$.

Direction cosines: $\cos\alpha = \dfrac{3}{5\sqrt{2}} = \dfrac{3\sqrt{2}}{10}$,

$\cos\beta = -\dfrac{4}{5\sqrt{2}} = -\dfrac{2\sqrt{2}}{5}$ and $\cos\gamma = \dfrac{5}{5\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

Example 3 Find the angles that $\mathbf{s} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$ makes with the axes.

$|\mathbf{s}| = \sqrt{6}$, $\hat{\mathbf{s}} = \dfrac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = \dfrac{1}{\sqrt{6}}\mathbf{i} - \dfrac{2}{\sqrt{6}}\mathbf{j} - \dfrac{1}{\sqrt{6}}\mathbf{k}$.

$\therefore \cos\alpha = \dfrac{1}{\sqrt{6}}$, $\cos\beta = -\dfrac{2}{\sqrt{6}}$, and $\cos\gamma = -\dfrac{1}{\sqrt{6}}$.

Hence $\alpha \approx 66^o$, $\beta \approx 145^o$ and $\gamma \approx 114^o$.

Example 4 $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three orthogonal (i.e. mutually perpendicular) vectors. Find an expression for the cosine of the angle between $(\mathbf{a} + \mathbf{b} + \mathbf{c})$ and $\mathbf{c}$.

Consider the scalar product of two vectors: $\mathbf{p}.\mathbf{q} = pq\cos\theta$,

$\therefore (\mathbf{a} + \mathbf{b} + \mathbf{c}).\mathbf{c} = |\mathbf{a} + \mathbf{b} + \mathbf{c}|\,|\mathbf{c}|\cos\theta$,

$\therefore \mathbf{a}.\mathbf{c} + \mathbf{b}.\mathbf{c} + \mathbf{c}.\mathbf{c} = \sqrt{(\mathbf{a} + \mathbf{b} + \mathbf{c}).(\mathbf{a} + \mathbf{b} + \mathbf{c})}\ c\cos\theta$.

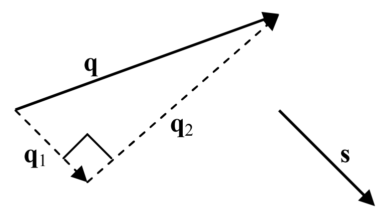
Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are orthogonal, $\therefore \mathbf{a}.\mathbf{b} = 0$, $\mathbf{b}.\mathbf{c} = 0$ and $\mathbf{c}.\mathbf{a} = 0$.

Hence $c^2 = \sqrt{a^2+b^2+c^2}\,c\cos\theta$, and

$\therefore \cos\theta = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$.

Scalar and vector resolutes

Instead of resolving a vector into $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ components, it can be resolved in other ways, e.g. vector $\mathbf{q}$ can be resolved into two perpendicular components, one component $\mathbf{q}_1$ parallel to another vector $\mathbf{s}$ and the other component $\mathbf{q}_2 \perp$ to $\mathbf{s}$.



q_1 and $\mathbf{q}_1$ are respectively called the scalar and vector resolutes of $\mathbf{q}$ parallel to $\mathbf{s}$ (or the scalar and vector projections of $\mathbf{q}$ onto $\mathbf{s}$).

q_2 and $\mathbf{q}_2$ are respectively called the scalar and vector resolutes of $\mathbf{q}$ perpendicular to $\mathbf{s}$.

$q_1 = \mathbf{q}.\hat{\mathbf{s}}$ and $\mathbf{q}_1 = q_1\hat{\mathbf{s}} = (\mathbf{q}.\hat{\mathbf{s}})\hat{\mathbf{s}}$,

$\mathbf{q}_2 = \mathbf{q} - \mathbf{q}_1$, and $q_2 = |\mathbf{q} - \mathbf{q}_1|$.

Example 1 Find the projection of $4\mathbf{i} + \mathbf{k}$ onto $-3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, i.e. scalar resolute of $4\mathbf{i} + \mathbf{k}$ in the direction of $-3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Let $\mathbf{h} = 4\mathbf{i} + \mathbf{k}$ and $\mathbf{g} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Unit vector in the direction of $\mathbf{g}$: $\hat{\mathbf{g}} = \dfrac{1}{\sqrt{14}}(-3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$.

The projection of $\mathbf{h}$ onto $\mathbf{g}$:

$\mathbf{h}.\hat{\mathbf{g}} = (4\mathbf{i} + \mathbf{k}).\dfrac{1}{\sqrt{14}}(-3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \dfrac{1}{\sqrt{14}}(-12 - 2) = -\sqrt{14}$.

Example 2 Find the scalar and vector resolutes of $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in the direction of $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$ and, perpendicular to $\mathbf{a}$.

$\hat{\mathbf{a}} = \dfrac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j})$.

Scalar resolute of **b** parallel to **a**: $\mathbf{b}.\hat{\mathbf{a}} = \dfrac{1}{\sqrt{13}}(2+6) = \dfrac{8}{\sqrt{13}}$.

Vector resolute of **b** parallel to **a**:

$(\mathbf{b}.\hat{\mathbf{a}})\hat{\mathbf{a}} = \dfrac{8}{\sqrt{13}} \times \dfrac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j}) = \dfrac{8}{13}(2\mathbf{i} - 3\mathbf{j})$.

Vector resolute of **b** perpendicular to **a**:

$\mathbf{b} - (\mathbf{b}.\hat{\mathbf{a}})\hat{\mathbf{a}} = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - \dfrac{8}{13}(2\mathbf{i} - 3\mathbf{j}) = -\dfrac{3}{13}\mathbf{i} - \dfrac{2}{13}\mathbf{j} + \mathbf{k}$.

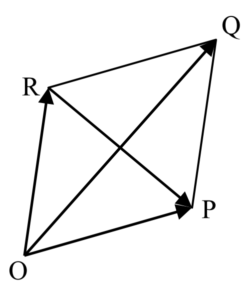
Scalar resolute of **b** perpendicular to **a**:

$|\mathbf{b} - (\mathbf{b}.\hat{\mathbf{a}})\hat{\mathbf{a}}| = \sqrt{\left(-\dfrac{3}{13}\right)^2 + \left(-\dfrac{2}{13}\right)^2 + 1^2} = \dfrac{\sqrt{182}}{13}$.

Vector proofs of simple geometric results

Example 1 Prove that the diagonals of a rhombus are perpendicular.

Consider rhombus OPQR, $\overline{OP} = \overline{PQ} = \overline{QR} = \overline{OR}$.



$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OR}$, $\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$,

$\overrightarrow{OQ} \bullet \overrightarrow{RP} = \left(\overrightarrow{OP} + \overrightarrow{OR}\right) \bullet \left(\overrightarrow{OP} - \overrightarrow{OR}\right)$

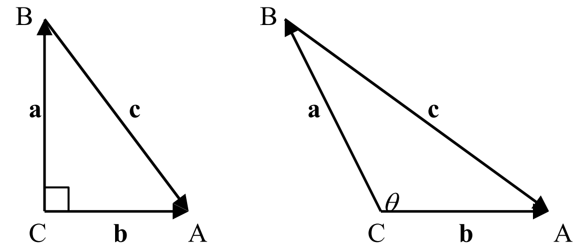
$= \overrightarrow{OP} \bullet \overrightarrow{OP} - \overrightarrow{OP} \bullet \overrightarrow{OR} + \overrightarrow{OR} \bullet \overrightarrow{OP} - \overrightarrow{OR} \bullet \overrightarrow{OR}$

$= \overline{OP}^2 - \overline{OR}^2 = 0$.

Since $\overrightarrow{OQ}$ and $\overrightarrow{RP}$ are non-zero vectors, and $\overrightarrow{OQ} \bullet \overrightarrow{RP} = 0$,

$\therefore$ the diagonals are perpendicular.

Example 2 Use vectors to prove Pythagoras' theorem and the cosine rule.



Introduce vectors **a**, **b** and **c** as shown above to simplify the notations.

Pythagoras' theorem: $\mathbf{c} = \mathbf{b} - \mathbf{a}$, $\mathbf{c}.\mathbf{c} = (\mathbf{b} - \mathbf{a}).(\mathbf{b} - \mathbf{a})$,

$\therefore \mathbf{c}.\mathbf{c} = \mathbf{b}.\mathbf{b} - \mathbf{b}.\mathbf{a} - \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{a}$.

Since $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a} = 0$, $\therefore c^2 = a^2 + b^2$.

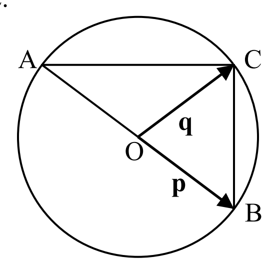
The cosine rule: $\mathbf{c} = \mathbf{b} - \mathbf{a}$, $\mathbf{c}.\mathbf{c} = (\mathbf{b} - \mathbf{a}).(\mathbf{b} - \mathbf{a})$,

$\therefore \mathbf{c}.\mathbf{c} = \mathbf{b}.\mathbf{b} - \mathbf{b}.\mathbf{a} - \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{a}$.

Since $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a} = ab\cos\theta = ab\cos C$,

$\therefore c^2 = a^2 + b^2 - 2ab\cos C$.

Example 3 Prove that the angle subtended by a diameter in a circle is a right angle.



Introduce vectors **p** and **q** from centre O to B and C respectively.

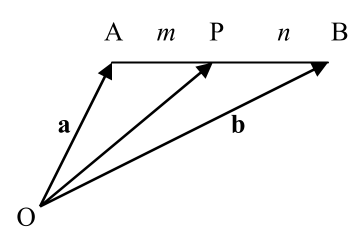
$\overrightarrow{AO} = \mathbf{p}$, $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \mathbf{p} + \mathbf{q}$, $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \mathbf{p} - \mathbf{q}$,

$\therefore \overrightarrow{AC} \bullet \overrightarrow{CB} = (\mathbf{p} + \mathbf{q}).(\mathbf{p} - \mathbf{q}) = \mathbf{p}.\mathbf{p} - \mathbf{q}.\mathbf{q} = p^2 - q^2 = 0$,

because $p = q =$ radius of the circle.

Hence $\overrightarrow{AC} \perp \overrightarrow{CB}$, i.e. $\angle ACB = 90^o$.

Example 4 Point P divides $\overline{AB}$ according to the ratio $m : n$. Find vector $\overrightarrow{OP}$ in terms of m, n, **a** and **b**.

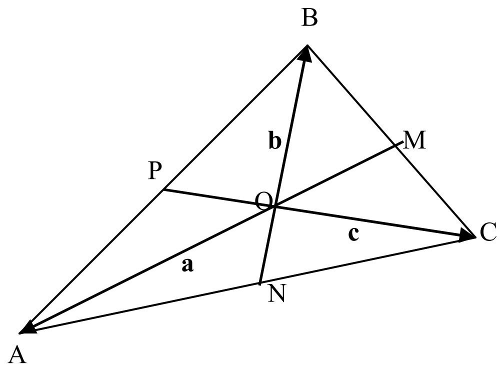


$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + \dfrac{m}{m+n}\overrightarrow{AB}$

$= \mathbf{a} + \dfrac{m}{m+n}(\mathbf{b} - \mathbf{a}) = \dfrac{1}{m+n}(m\mathbf{a} + n\mathbf{a} + m\mathbf{b} - m\mathbf{a})$

$= \dfrac{1}{m+n}(n\mathbf{a} + m\mathbf{b})$.

Example 5 Prove that the medians of a triangle are concurrent, and the intersection trisects each median.



$\overline{AM}$ and $\overline{BN}$ are medians and $\overline{CP}$ is a line segment passing through the intersection O of $\overline{AM}$ and $\overline{BN}$.

Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. $\therefore \overrightarrow{OM} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$ and $\overrightarrow{ON} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$. Let m, n and p be some positive constants such that $\overrightarrow{OM} = -m\,\mathbf{a}$, $\overrightarrow{ON} = -n\,\mathbf{b}$ and $\overrightarrow{OP} = -p\,\mathbf{c}$.

$\therefore -m\,\mathbf{a} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$ and $-n\,\mathbf{b} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$.

From the last two equations, $-2m\mathbf{a} - \mathbf{b} = -2n\mathbf{b} - \mathbf{a}$,
hence $(1-2m)\mathbf{a} - (1-2n)\mathbf{b} = 0$.
Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, $\therefore$ they are independent and hence $1-2m = 0$ and $-(1-2n) = 0$, i.e. $m = n = \frac{1}{2}$.

$\therefore -\frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$, $\therefore \mathbf{b} = -\mathbf{a} - \mathbf{c}$.

$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = -\mathbf{a} - p\mathbf{c}$, and $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = -2\mathbf{a} - \mathbf{c}$.

Let $\overrightarrow{AP} = k\overrightarrow{AB}$ where k is a positive constant.

$\therefore$ $-\mathbf{a} - p\mathbf{c} = k(-2\mathbf{a} - \mathbf{c})$, i.e. $(2k-1)\mathbf{a} + (k-p)\mathbf{c} = 0$.

Since $\mathbf{a}$ and $\mathbf{c}$ are independent, $\therefore 2k-1 = 0$ and $k - p = 0$, i.e. $k = \frac{1}{2}$ and $p = \frac{1}{2}$. $\therefore$ P is the mid-point of $\overline{AB}$ and $\overline{CP}$ is a median. Hence the three medians are concurrent at O and the intersection O trisects each median.

Vector equations, parametric equations and cartesian equations (2-dimensional)

As a particle moves its position changes with time. Its position can be described with a position vector $\mathbf{r}(t)$ that is expressed in terms of $\mathbf{i}$ and $\mathbf{j}$ components, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.

This equation is called a vector equation and t the parameter.

$x = x(t)$ and $y = y(t)$ are two rectangular components in terms of the parameter t. They are called parametric equations.

When the parameter t is eliminated from the two parametric equations, we obtain a cartesian equation of the locus (path), $f(x,y) = 0$.

Example 1 Sketch the locus of a particle with its position described by $\mathbf{r}(t) = t^2\,\mathbf{i} - (t-1)\mathbf{j},\ t \ge 0$.

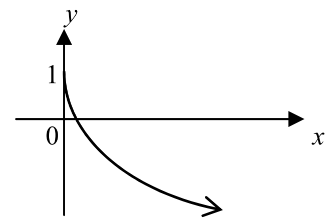
Vector equation: $\mathbf{r}(t) = t^2\,\mathbf{i} - (t-1)\mathbf{j},\ t \ge 0$.

Parametric equations: $x = t^2,\ y = -(t-1)$.

Cartesian equation:

From the second equation, $t = 1 - y$, substitute into the first equation, $x = (1-y)^2$, $\therefore y = \pm\sqrt{x} + 1$

Since $t \ge 0$, $\therefore y = -(t-1) \le 1$, $\therefore y = -\sqrt{x} + 1$.



Example 2 Sketch the locus of a particle with position vector $\mathbf{r}(t) = (3\sin t + 1)\,\mathbf{i} + (2\cos t - 1)\,\mathbf{j}$, and $t \ge 0$.

Vector equation: $\mathbf{r}(t) = (3\sin t + 1)\,\mathbf{i} + (2\cos t - 1)\,\mathbf{j}$, and $t \ge 0$.

Parametric equations:

$x = 3\sin t + 1,\ \therefore \sin t = \frac{x-1}{3}$,

$y = 2\cos t - 1,\ \therefore \cos t = \frac{y+1}{2}$.

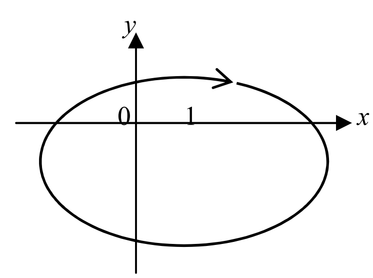
Cartesian equation:

Since $\sin^2 t + \cos^2 t = 1$, $\therefore \left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{2}\right)^2 = 1$,

i.e. $\frac{(x-1)^2}{9} + \frac{(y+1)^2}{4} = 1$.

The locus is an ellipse centred at $(1,-1)$. At $t = 0$, the particle is at $x = 1,\ y = 1$. At $t = \frac{\pi}{6}$, it is at $x = 2.5,\ y = \sqrt{3} - 1$.

$\therefore$ the particle starts at $(1,1)$ and moves clockwise.



Example 3 Sketch the locus of a particle with position given by $\mathbf{r}(t) = \left(t+\frac{1}{t}\right)\mathbf{i} + \left(t-\frac{1}{t}\right)\mathbf{j}$, $t > 0$.

Vector equation: $\mathbf{r}(t) = \left(t+\frac{1}{t}\right)\mathbf{i} + \left(t-\frac{1}{t}\right)\mathbf{j}$, $t > 0$.

Parametric equations:

$x = t+\frac{1}{t}$ ……(1), $y = t-\frac{1}{t}$ ……(2)

Cartesian equation:

(1) + (2), $x + y = 2t$, $\therefore t = \frac{x+y}{2}$.

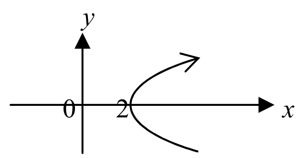
Substitute into (1), $x = \frac{x+y}{2} + \frac{2}{x+y}$,

$\therefore x = \frac{(x+y)^2+4}{2(x+y)}$, $\therefore 2x(x+y) = (x+y)^2+4$,

$2x^2+2xy = x^2+2xy+y^2+4$, $\therefore x^2-y^2 = 4$,

i.e. $\frac{x^2}{4}-\frac{y^2}{4} = 1$.

The locus of the particle is a hyperbola.
For $t > 0$, $x > 0$, $\therefore$ the locus consists of the right hand branch only. At $t \to 0$, $x \to +\infty$, $y \to -\infty$, $\therefore$ the particle moves upwards along the hyperbola.



Example 4 A particle has position vector $\mathbf{r}(t) = \left(t^2+\frac{1}{t^2}\right)\mathbf{i} + \left(t^2-\frac{1}{t^2}\right)\mathbf{j}$, $t > 0$. Find the cartesian equation and sketch the graph of its locus. Compare the motion of this particle with that in example 3.

Vector equation: $\mathbf{r}(t) = \left(t^2+\frac{1}{t^2}\right)\mathbf{i} + \left(t^2-\frac{1}{t^2}\right)\mathbf{j}$, $t > 0$.

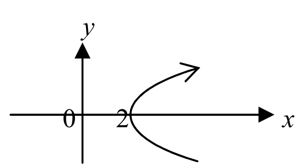
Parametric equations:

$x = t^2+\frac{1}{t^2}$ ……(1), $y = t^2-\frac{1}{t^2}$ ……(2)

Cartesian equation:

(1) + (2), $x + y = 2t^2$, $\therefore t^2 = \frac{x+y}{2}$.

Substitute into (1), $x = \frac{x+y}{2} + \frac{2}{x+y}$, $\therefore \frac{x^2}{4}-\frac{y^2}{4} = 1$.



The particle has the same locus but different speed.

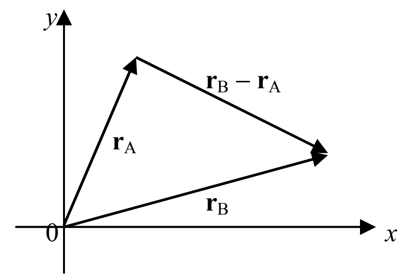
Example 5 Two ships are on a collision course. Their position vectors are $\mathbf{r}(t) = (2t + 11)\mathbf{i} + (7t + 6)\mathbf{j}$ and $\mathbf{r}(t) = (5t + 7)\mathbf{i} + (4t + 10)\mathbf{j}$, where $t \ge 0$. When and where do the ships collide?

The two ships collide when they are at the same place and at the same time. Let $(2t + 11)\mathbf{i} + (7t + 6)\mathbf{j} = (5t + 7)\mathbf{i} + (4t + 10)\mathbf{j}$.
Equate corresponding components:

$2t + 11 = 5t + 7$ and $7t + 6 = 4t + 10$, $\therefore t = \frac{4}{3}$.

At $t = \frac{4}{3}$, the ships are at the same position $\mathbf{r}(t) = \frac{41}{3}\mathbf{i} + \frac{46}{3}\mathbf{j}$.

Example 6 The position vectors for particles A and B are $\mathbf{r}_A(t) = (2t^2+t-1)\,\mathbf{i} + (2t+3)\,\mathbf{j}$ and $\mathbf{r}_B(t) = (t^2+5t-4)\,\mathbf{i} + 3t\mathbf{j}$ respectively. Find an expression for their separation at time $t \ge 0$. When and where do they collide?



$\mathbf{r}_B - \mathbf{r}_A = (t^2+5t-4)\,\mathbf{i} + 3t\mathbf{j} - [\,(2t^2+t-1)\,\mathbf{i} + (2t+3)\,\mathbf{j}]$

$= -(t^2-4t+3)\,\mathbf{i} + (t-3)\,\mathbf{j}$

Separation = $|\mathbf{r}_B - \mathbf{r}_A| = \sqrt{(t^2-4t+3)^2+(t-3)^2}$.

The particles collide when their separation is zero, i.e. $\sqrt{(t^2-4t+3)^2+(t-3)^2} = 0$, $(t^2-4t+3)^2+(t-3)^2 = 0$,

$(t-3)^2(t-1)^2+(t-3)^2 = 0$, $(t-3)^2((t-1)^2+1) = 0$,

$\therefore t = 3$ and hence $\mathbf{r} = 20\mathbf{i} + 9\mathbf{j}$ is where they collide.

Differentiation of a vector with respect to time

Given vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, its first and second derivatives with respect to t are respectively

$$\frac{d}{dt}\mathbf{r} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \text{ and } \frac{d^2}{dt^2}\mathbf{r} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}.$$

Derivatives of position and velocity vectors

If $\mathbf{r}(t)$ is a position vector, its first derivative is a velocity vector $\mathbf{v}(t)$, and the second derivative is an acceleration vector $\mathbf{a}(t)$.

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r},\ \mathbf{a}(t) = \frac{d}{dt}\mathbf{v} \text{ or } \mathbf{a}(t) = \frac{d^2}{dt^2}\mathbf{r}.$$

Example 1 The position vector of an object moving in a plane is given by $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j},\ t \ge 0$. (a) Find its velocity, speed and acceleration when $t = 1$. (b) Draw the graph of the locus, and the velocity and acceleration vectors at $t = 1$.

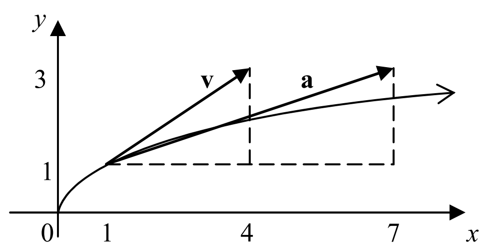
(a) $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j},\ t \ge 0$, $\mathbf{v}(t) = \dfrac{d}{dt}\mathbf{r} = 3t^2\,\mathbf{i} + 2t\,\mathbf{j}$,

$v = \sqrt{\left(3t^2\right)^2 + \left(2t\right)^2} = \sqrt{9t^4 + 4t^2}$,

$\mathbf{a}(t) = \dfrac{d}{dt}\mathbf{v} = 6t\,\mathbf{i} + 2\mathbf{j}$.

At $t = 1$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$, $v = \sqrt{13}$, $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j}$.

(b) $x = t^3$, $y = t^2$, $\therefore y = \left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$. Since $t \ge 0$, $\therefore x \ge 0$.



Example 2 Find the velocity, acceleration and speed of a particle whose position vector is given by $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$, $t \ge 0$. Sketch the path of the particle and draw the velocity and acceleration vectors at $t = 0$.

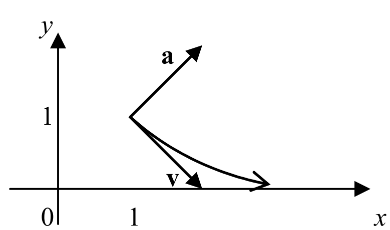
$\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$, $\mathbf{v}(t) = \dfrac{d}{dt}\mathbf{r} = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$,

$v = \sqrt{\left(e^t\right)^2 + \left(-e^{-t}\right)^2} = \sqrt{e^{2t} + e^{-2t}}$, $\mathbf{a}(t) = \dfrac{d}{dt}\mathbf{v} = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$.

At $t = 0$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{a} = \mathbf{i} + \mathbf{j}$.

Parametric equations: $x = e^t$, $y = e^{-t}$.

Cartesian equation: $y = e^{-t} = \dfrac{1}{e^t}$, $\therefore y = \dfrac{1}{x}$. When $t = 0$, $x = 1$ and $y = 1$. $\therefore$ the particle starts from $(1,1)$.



Example 3 A particle moves so that its position vector at time t is given by $\mathbf{r} = 3\cos(2t)\mathbf{i} + 3\sin(2t)\mathbf{j}$ for $t \ge 0$.
(a) Find the velocity and acceleration at time t.
(b) Show that in this case, $\mathbf{v}$ is perpendicular to $\mathbf{r}$, and $\mathbf{a}$ is opposite in direction to $\mathbf{r}$.
(c) Sketch the locus of the particle and show the velocity and acceleration vectors at time t.

(a) $\mathbf{r} = 3\cos(2t)\mathbf{i} + 3\sin(2t)\mathbf{j}$,

$\mathbf{v} = \dfrac{d}{dt}\mathbf{r} = -6\sin(2t)\,\mathbf{i} + 6\cos(2t)\,\mathbf{j}$,

$\mathbf{a} = \dfrac{d}{dt}\mathbf{v} = -12\cos(2t)\,\mathbf{i} - 12\sin(2t)\,\mathbf{j}$.

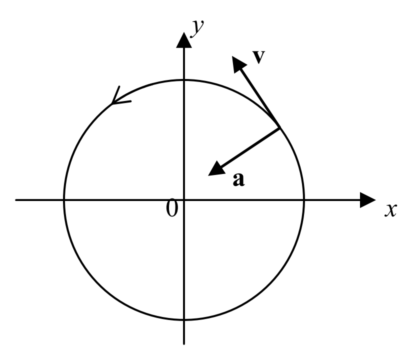
(b) $\mathbf{v.r} = -18\sin(2t)\cos(2t) + 18\sin(2t)\cos(2t) = 0$, $\therefore \mathbf{v} \perp \mathbf{r}$.
$\mathbf{a} = -12\cos(2t)\,\mathbf{i} - 12\sin(2t)\,\mathbf{j} = -4(3\cos(2t)\,\mathbf{i} + 3\sin(2t)\,\mathbf{j})$,
$\therefore \mathbf{a} = -4\,\mathbf{r}$. Hence $\mathbf{a}$ is opposite in direction to $\mathbf{r}$.

(c) $x = 3\cos(2t)$, $\therefore \dfrac{x}{3} = \cos(2t)$. $y = 3\sin(2t)$, $\therefore \dfrac{y}{3} = \sin(2t)$.

Use $\sin^2 A + \cos^2 A = 1$ to eliminate t: $\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 1$,

$\therefore x^2 + y^2 = 9$. The locus is a circle of radius 3 and centre $(0,0)$. At $t = 0$, $x = 3\cos 0 = 3$, $y = 3\sin 0 = 0$.

At $t = 0.1$, $x > 0$ and $y > 0$. $\therefore$ The particle starts from $(3,0)$ and moves anticlockwise.



Example 4 An object moves in a path with position vector $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $t \ge 0$. (a) Show that the path is elliptical. (b) Show that the acceleration points towards the origin.

(a) $x = \cos t$, $y = 2\sin t$, $\therefore \dfrac{y}{2} = \sin t$.

Eliminate t, $x^2 + \dfrac{y^2}{4} = 1$. Path is elliptical.

(b) $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $\mathbf{v} = -\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$,
$\mathbf{a} = -\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} = -(\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}) = -\mathbf{r}$. $\therefore \mathbf{a}$ is opposite to $\mathbf{r}$ and hence it is towards the origin.

Antidifferentiation of a vector with respect to time

Given acceleration $\mathbf{a}(t) = p(t)\mathbf{i} + q(t)\mathbf{j}$,

velocity is $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int p(t)\,\mathbf{i}\,dt + \int q(t)\,\mathbf{j}\,dt$, and

position is $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt$.

Example 1 Given the acceleration $\mathbf{a} = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, and when $t = 0$, $\mathbf{r} = -2\mathbf{i}$ and $\mathbf{v} = -2\mathbf{j}$, of a particle.

(a) Show that the speed of the particle is constant.
(b) Show that the velocity and the acceleration are always perpendicular.
(c) Show that the acceleration is always towards the centre of the circular path.
(d) Show that the path is circular.
(e) Sketch the path, and show the direction of motion of the particle at time t.

(a) $\mathbf{a} = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $\mathbf{v} = \int 2\cos t\,\mathbf{i}\,dt + \int 2\sin t\,\mathbf{j}\,dt$,

$\therefore \mathbf{v} = 2\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j} + \mathbf{c}$, where $\mathbf{c}$ is a constant vector.

When $t = 0$, $\mathbf{v} = -2\mathbf{j}$, $\therefore \mathbf{c} = \mathbf{0}$, $\therefore\ \mathbf{v} = 2\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j}$.

Hence $v = \sqrt{(2\sin t)^2 + (-2\cos t)^2} = \sqrt{4} = 2$, a constant.

(b) $\mathbf{v}.\mathbf{a} = 4\sin t\cos t - 4\cos t\sin t = 0$, $\therefore \mathbf{v}$ is $\perp$ $\mathbf{a}$.

(c) $\mathbf{r} = \int 2\sin t\,\mathbf{i}\,dt - \int 2\cos t\,\mathbf{j}\,dt = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} + \mathbf{d}$,
where $\mathbf{d}$ is a constant vector.

When $t = 0$, $\mathbf{r} = -2\mathbf{i}$, $\therefore \mathbf{d} = \mathbf{0}$, $\therefore \mathbf{r} = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$.

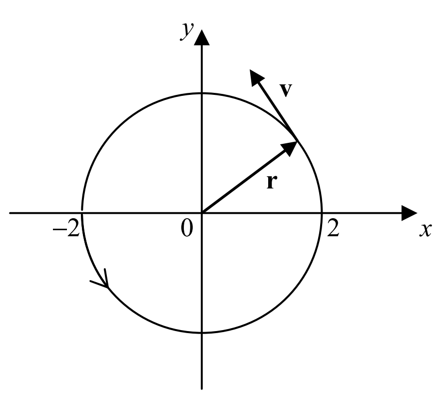
$\therefore \mathbf{a} = -\mathbf{r}$. Hence $\mathbf{a}$ is always towards the origin, i.e. the centre of the circular path.

(d) $x = -2\cos t$, $\therefore -\dfrac{x}{2} = \cos t$. $y = -2\sin t$, $\therefore -\dfrac{y}{2} = \sin t$.

Use $\sin^2 A + \cos^2 A = 1$ to eliminate t: $\left(-\dfrac{x}{2}\right)^2 + \left(-\dfrac{y}{2}\right)^2 = 1$,

$\therefore x^2 + y^2 = 4$. The path is a circle of radius 2 and centre $(0,0)$.

(e)



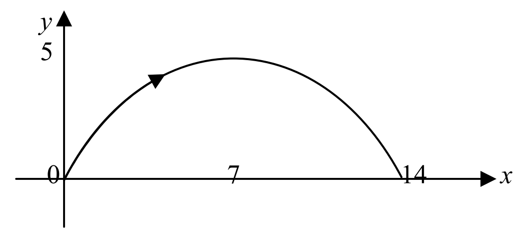
Example 2 Given the acceleration $\mathbf{a} = -10\mathbf{j}$, and when $t = 0$, the position is $\mathbf{r} = \mathbf{0}$ and the velocity is $\mathbf{v} = 7\mathbf{i} + 10\mathbf{j}$.
(a) Find the position vector at time $t \geq 0$.
(b) Describe the path.
(c) At what time is the y-coordinate the same as that at $t = 0$?
(d) What is the x-coordinate at that time?

(a) $\mathbf{a} = -10\mathbf{j}$, $\mathbf{v} = \int -10\,\mathbf{j}\,dt = -10t\,\mathbf{j} + \mathbf{c}$, $\mathbf{c}$ is a constant vector.
When $t = 0$, the velocity is $\mathbf{v} = 7\mathbf{i} + 10\mathbf{j}$, $\therefore \mathbf{c} = 7\mathbf{i} + 10\mathbf{j}$.
Hence $\mathbf{v} = -10t\,\mathbf{j} + 7\mathbf{i} + 10\mathbf{j} = 7\mathbf{i} + (10 - 10t)\,\mathbf{j}$.
$\therefore \mathbf{r} = \int 7\,\mathbf{i}\,dt + \int (10 - 10t)\,\mathbf{j}\,dt = 7t\,\mathbf{i} + (10t - 5t^2)\,\mathbf{j} + \mathbf{d}$, where $\mathbf{d}$ is a constant vector.
When $t = 0$, the position $\mathbf{r} = \mathbf{0}$, $\therefore\ \mathbf{d} = \mathbf{0}$.
Hence $\mathbf{r} = 7t\,\mathbf{i} + (10t - 5t^2)\,\mathbf{j}$.

(b) $x = 7t$, $y = 10t - 5t^2$.

Eliminate t, $y = 10\left(\dfrac{x}{7}\right) - 5\left(\dfrac{x}{7}\right)^2$, $y = \dfrac{10}{7}x - \dfrac{5}{49}x^2$,

$y = -\dfrac{5}{49}\left(x^2 - 14x\right) = -\dfrac{5}{49}\left(x^2 - 14x + 49\right) + 5 = -\dfrac{5}{49}(x - 7)^2 + 5$.
Path is a parabola, vertex at (7,5).



(c) At $t = 0$, $y = 0$. When $y = 0$, $10t - 5t^2 = 0$, $\therefore t = 2$.

(d) At $t = 2$, $x = 2t = 14$.

Example 3 A particle is projected at the origin with a velocity given by $\mathbf{v} = \mathbf{i} + \mathbf{j} + 10\mathbf{k}$, and it has a constant acceleration $\mathbf{a} = -10\mathbf{k}$. The unit vector $\mathbf{k}$ is pointing vertically upward.

(a) Find the velocity vector of the particle at time t.
(b) Find the position vector of the particle at time t.
(c) Find the maximum height above the origin.
(d) Find the distance of the particle from the origin when it returns to the same level as the origin.

(a) $\mathbf{a} = -10\mathbf{k}$, $\mathbf{v} = \int -10\,\mathbf{k}\,dt = -10t\,\mathbf{k} + \mathbf{c}$, where $\mathbf{c}$ is a constant vector.
When $t = 0$, the velocity is $\mathbf{v} = \mathbf{i} + \mathbf{j} + 10\mathbf{k}$, $\therefore \mathbf{c} = \mathbf{i} + \mathbf{j} + 10\mathbf{k}$.
Hence $\mathbf{v} = -10t\,\mathbf{k} + \mathbf{i} + \mathbf{j} + 10\mathbf{k} = \mathbf{i} + \mathbf{j} + (10 - 10t)\,\mathbf{k}$.

(b) $\therefore \mathbf{r} = \int \mathbf{i}\,dt + \int \mathbf{j}\,dt + \int (10 - 10t)\,\mathbf{k}\,dt$

$= t\,\mathbf{i} + t\,\mathbf{j} + (10t - 5t^2)\,\mathbf{k} + \mathbf{d}$, where $\mathbf{d}$ is a constant vector.
When $t = 0$, the position $\mathbf{r} = \mathbf{0}$, $\therefore\ \mathbf{d} = \mathbf{0}$.
Hence $\mathbf{r} = t\,\mathbf{i} + t\,\mathbf{j} + (10t - 5t^2)\,\mathbf{k}$.

(c) At the highest point, the particle moves horizontally, i.e. the **k** component of **v** is zero. $\therefore 10-10t=0,\ t=1$.

At $t=1$, the **k** component of **r** is $10t-5t^2=10-5=5$, $\therefore$ the maximum height is 5.

(d) When the particle returns to the starting level, the **k** component of **r** is zero. $\therefore 10t-5t^2=0,\ t=2$.

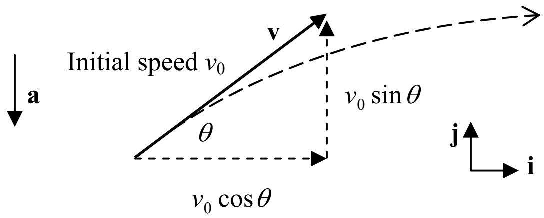
At $t=2$, $\mathbf{r}=t\,\mathbf{i}+t\,\mathbf{j}+(10t-5t^2)\,\mathbf{k}=2\mathbf{i}+2\mathbf{j}$.

$\therefore$ distance from origin $=|\mathbf{r}(2)|=\sqrt{2^2+2^2}=2\sqrt{2}$.

The last two examples are projectile motions under constant acceleration due to gravity, g.

A general consideration of projectile motion under constant acceleration due to gravity (2 D)

Let $\mathbf{a}=-g\mathbf{j}$ and the initial position and velocity are respectively $\mathbf{r}=\mathbf{0}$ and $\mathbf{v}=v_0\cos\theta\,\mathbf{i}+v_0\sin\theta\,\mathbf{j}$.



$\mathbf{a}=-g\mathbf{j}$, $\mathbf{v}=\int -g\,\mathbf{j}\,dt=-gt\,\mathbf{j}+\mathbf{c}$, where **c** is a constant vector.

When $t=0$, the velocity is $\mathbf{v}=v_0\cos\theta\,\mathbf{i}+v_0\sin\theta\,\mathbf{j}$,

$\therefore \mathbf{c}=v_0\cos\theta\,\mathbf{i}+v_0\sin\theta\,\mathbf{j}$.

Hence $\mathbf{v}=-gt\,\mathbf{j}+v_0\cos\theta\,\mathbf{i}+v_0\sin\theta\,\mathbf{j}$

$=v_0\cos\theta\,\mathbf{i}+(v_0\sin\theta-gt)\,\mathbf{j}$.

$\therefore \mathbf{r}=\int v_0\cos\theta\,\mathbf{i}\,dt+\int(v_0\sin\theta-gt)\,\mathbf{j}\,dt$

$=(v_0\cos\theta)t\,\mathbf{i}+((v_0\sin\theta)t-\frac{1}{2}gt^2)\,\mathbf{j}+\mathbf{d}$, where **d** is a constant vector.

When $t=0$, the position $\mathbf{r}=\mathbf{0}$, $\therefore\ \mathbf{d}=\mathbf{0}$.

$\therefore\ \mathbf{r}=(v_0\cos\theta)t\,\mathbf{i}+((v_0\sin\theta)t-\frac{1}{2}gt^2)\,\mathbf{j}$.

Hence at time t, the horizontal and vertical components of the velocity vector are $v=v_0\cos\theta$ and $v=v_0\sin\theta-gt$ respectively.

Also, at time t, the horizontal and vertical components of the position vector are $x=(v_0\cos\theta)t$ and $y=(v_0\sin\theta)t-\frac{1}{2}gt^2$ respectively.

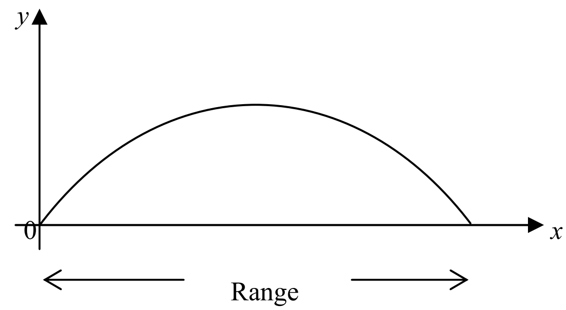
Eliminating t from the last two equations:

$x=(v_0\cos\theta)t$, $\therefore t=\frac{x}{v_0\cos\theta}$. Substitute into

$y=(v_0\sin\theta)t-\frac{1}{2}gt^2$,

$y=(v_0\sin\theta)\frac{x}{v_0\cos\theta}-\frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2$,

$\therefore y=x\tan\theta-\frac{g}{2(v_0\cos\theta)^2}x^2$. This is the equation of the locus of the projectile satisfying the conditions stated in the opening sentence.



The maximum height is reached when the **j** component of **v** is zero, i.e. $v_0\sin\theta-gt=0$, $\therefore t=\frac{v_0\sin\theta}{g}$.

At $t=\frac{v_0\sin\theta}{g}$,

the **j** component of **r** is $y=(v_0\sin\theta)t-\frac{1}{2}gt^2=\frac{(v_0\sin\theta)^2}{2g}$.

$\therefore$ maximum height $=\frac{(v_0\sin\theta)^2}{2g}$.

Also, maximum height is reached at mid-range where

$x=(v_0\cos\theta)t=(v_0\cos\theta)\frac{v_0\sin\theta}{g}=\frac{v_0^2\sin\theta\cos\theta}{g}=\frac{v_0^2\sin2\theta}{2g}$.

$\therefore$ the range of the projectile is $\frac{v_0^2\sin2\theta}{g}$.

Example 1 A particle is projected with a speed of 20 $\mathrm{ms^{-1}}$ at 60^o with the horizontal from the origin. Take $g=10\ \mathrm{ms^{-2}}$.

(a) Find the equation of the path of the particle.
(b) Find the maximum height reached.
(c) Find the range of the particle.

(a) $y=x\tan60^o-\frac{10}{2(20\cos60^o)^2}x^2$, $\therefore y=\sqrt{3}x-\frac{x^2}{20}$.

(b) Maximum height $=\frac{(v_0\sin\theta)^2}{2g}=\frac{(20\sin60^o)^2}{2\times10}=15$ m.

(c) Range $=\frac{v_0^2\sin2\theta}{g}=\frac{20^2\sin120^o}{10}=20\sqrt{3}$ m.