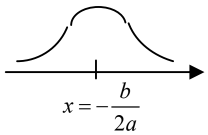
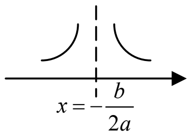
Specialist Mathematics Summary sheets

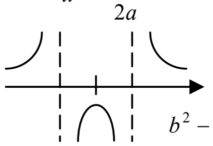
Coordinate geometry:

Graphs of $f(x)=\frac{1}{ax^2+bx+c}$, $a>0$

$b^2-4ac<0$ $\qquad$ $b^2-4ac=0$







$b^2-4ac>0$, equations of asymptotes are $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Graphs of ellipses: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

Centre (h,k), x-intercepts at $x=h\pm a\sqrt{1-\frac{k^2}{b^2}}$ for $1-\frac{k^2}{b^2}\ge 0$, y-intercepts at $y=k\pm b\sqrt{1-\frac{h^2}{a^2}}$ for $1-\frac{h^2}{a^2}\ge 0$.

Graphs of hyperbolas: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

Centre (h,k), x-intercepts at $x=h\pm a\sqrt{\frac{k^2}{b^2}+1}$,

y-intercepts at $y=k\pm b\sqrt{\frac{h^2}{a^2}-1}$ for $\frac{h^2}{a^2}-1\ge 0$.

Equations of asymptotes are $y=\pm\frac{b}{a}(x-h)+k$.

Transformations of trig. and inverse trig. functions: Express functions in the following forms before determining what transformations are required and always do translations last, e.g.

$y=-2\sec\left[-\frac{1}{2}(x+\pi)\right]+2$

6. Vertical translation (up 2)
5. Horizontal trans. (left π)
4. Horiz. dilation by 2 ×
2. Reflection in y-axis
3. Vertical dilation by 2 ×
1. Reflection in x-axis

e.g. $y=-3Cos^{-1}(1-2x)-\frac{3\pi}{2}$, express as

$y=-3Cos^{-1}\left[-2\left(x-\frac{1}{2}\right)\right]-\frac{3\pi}{2}$.

1 3 $\quad$ 2 4 $\quad$ 5 $\quad$ 6

Compound and double angles: $\frac{\pi}{12}=\frac{\pi}{3}-\frac{\pi}{4}$,

$\frac{5\pi}{12}=\frac{2\pi}{3}-\frac{\pi}{4}$, $\frac{7\pi}{12}=\frac{\pi}{3}+\frac{\pi}{4}$, $\frac{11\pi}{12}=\frac{2\pi}{3}+\frac{\pi}{4}$,

$\frac{\pi}{4}=2\left(\frac{\pi}{8}\right)$, $\frac{\pi}{6}=2\left(\frac{\pi}{12}\right)$, e.g. exact value $\cos\frac{5\pi}{12}$

$=\cos\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{2\pi}{3}\cos\frac{\pi}{4}+\sin\frac{2\pi}{3}\sin\frac{\pi}{4}$

$=-\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}=\frac{-1+\sqrt{3}}{2\sqrt{2}}$.

e.g. exact value of $\tan\frac{5\pi}{8}$: let $\frac{5\pi}{4}=2\left(\frac{5\pi}{8}\right)$,

$\tan\frac{5\pi}{4}=\tan 2\left(\frac{5\pi}{8}\right)$, $1=\frac{2\tan\frac{5\pi}{8}}{1-\tan^2\frac{5\pi}{8}}$,

$\therefore \tan^2\frac{5\pi}{8}+2\tan\frac{5\pi}{8}-1=0$, use the quadratic formula to find $\tan\frac{5\pi}{8}=-1\pm\sqrt{2}$ and since $\frac{5\pi}{8}$ is in the second quadrant, correct sol. is $-1-\sqrt{2}$.

Partial fractions:

e.g. $\frac{x+2}{x^2-3x-4}=\frac{x+2}{(x-4)(x+1)}=\frac{A}{x-4}+\frac{B}{x+1}$

$=\frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$, $x+2=A(x+1)+B(x-4)$

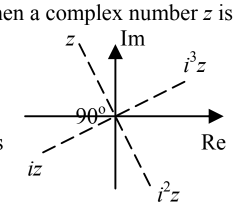
e.g. $\frac{2x-1}{x^2-2x+1}=\frac{2x-1}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$

$=\frac{A(x-1)+B}{(x-1)^2}$, $2x-1=A(x-1)+B$ etc.

If degree of numerator is equal to/ higher than that of denominator, do long division first, e.g.

$\frac{x^2-2x-2}{x^2-3x-4}=1+\frac{x+2}{x^2-3x-4}=1+\frac{A}{x-4}+\frac{B}{x+1}$

Complex numbers: When a complex number z is multiplied by i, i.e. iz is the rotation of z by 90° anticlockwise about O in the complex plane. Multiplying by $-i$ results in clockwise rotation of z by 90°.



$\frac{z}{i}=-iz$, clockwise 90°; $\frac{z}{-i}=iz$, anticlock 90°.

Division: $z_1\div z_2=\frac{z_1\times\overline{z_2}}{z_2\times\overline{z_2}}$,

e.g. $(1-i)\div(3+4i)=\frac{(1-i)(3-4i)}{(3+4i)(3-4i)}=\frac{-1-7i}{25}$.

Polar form: $z=rcis\theta$, $\operatorname{Re} z=r\cos\theta$, $\operatorname{Im} z=r\sin\theta$, $|z|=r$, $\arg z=\theta$, $\bar{z}=rcis(-\theta)$,

$z^{-1}=\frac{1}{r}cis(-\theta)$, $z^2=r^2cis2\theta$,

$\sqrt[n]{z}=z^{\frac{1}{n}}=r^{\frac{1}{n}}cis\left[\frac{1}{n}(\theta+2k\pi)\right]$.

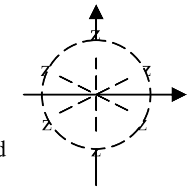
Example: $z=-\sqrt{2}cis\left(-\frac{\pi}{3}\right)$, find $|z|$ and Arg z.

$z=i^2\sqrt{2}cis\left(-\frac{\pi}{3}\right)$, $i^2\sqrt{2}cis\left(-\frac{\pi}{3}\right)$ is the anticlockwise rotation of $\sqrt{2}cis\left(-\frac{\pi}{3}\right)$ by 180°, $\therefore z=\sqrt{2}cis\left(-\frac{\pi}{3}+\pi\right)=\sqrt{2}cis\frac{2\pi}{3}$

$\therefore |z|=\sqrt{2}$, Arg $z=\frac{2\pi}{3}$.

Locating roots of a number in the complex plane: e.g. Plot the sixth roots of –1 in the complex plane.

There are 6 of them and the obvious one is i because $i^6=-1$. They are spaced out at equal angles on a circle centred at O and has a radius = | i | = 1.



Each angle $=\frac{360^o}{6}=60^o$.

Fundamental theorem of algebra: An n-degree polynomial (equation) always has n linear factors (roots,solutions) over C.

The conjugate root theorem: For a polynomial equation with real coefficients, the non-real roots always occur in conjugate pairs, e.g. $z^3+2z^2-6z+8=0$ is a third degree polynomial equation with real coefficients. $\therefore$ it has 3 roots according to the Fundamental theorem of algebra. Given that $1+i$ is one of the roots, $1-i$ must be another root according to the conjugate root theorem. The third root must be a real number otherwise a fourth one exists and contradicts the Fund. theorem of algebra. To find the third root, write the conjugate roots and the third root c as factors of the polynomial, z^3+2z^2-6z+8

$=(z-(1+i))(z-(1-i))(z-c)$

$=(z^2-2z+2)(z-c)$, $\therefore -2c=8$, $c=-4$.

Polynomials with complex coefficients:

e.g. Factorise z^2-i.

Let $z^2-i=(z-(a+bi))(z+(a+bi))$

$=z^2-(a^2-b^2)-2abi$.

$\therefore a^2-b^2=0$ and $2ab=1$. Solve to obtain $a=\frac{\sqrt{2}}{2}$ and $b=\frac{\sqrt{2}}{2}$.

e.g. Factorise $z^3-(2-i)z^2+z-2+i$

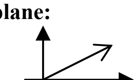
$=(z^3-(2-i)z^2)+(z-2+i)$

$=z^2(z-2+i)+1(z-2+i)$

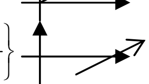
$=(z^2+1)(z-2+i)$ $=(z-i)(z+i)(z-2+i)$

Relations in the complex plane:

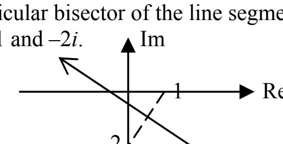
A ray, e.g. $\left\{z:Argz=\frac{\pi}{6}\right\}$



e.g. $\left\{z:Arg(z-(2-i))=\frac{\pi}{6}\right\}$

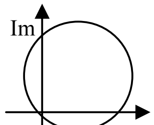


A line, e.g. $\{z:|z-1|=|z+2i|\}$ is the perpendicular bisector of the line segment joining 1 and $-2i$.



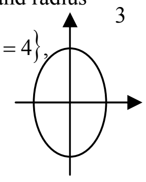
The above line has equation $2x+4y=-3$, can be defined as $\{z:2\operatorname{Re}z+4\operatorname{Im}z=-3\}$.

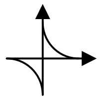
A circle, e.g. $\{z : |z-(1+i)| = \sqrt{2}\}$, centred at $z = 1+i$, radius $\sqrt{2}$.



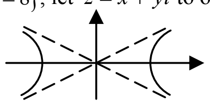
e.g. $\{z : 2|z+1| = |z-i|\}$ also defined a circle.
Let $z = x+yi$, $2|(x+1)+yi| = |x+(y-1)i|$,
square both sides, $4(x+1)^2+4y^2 = x^2+(y-1)^2$,
simplify to $\left(x+\frac{4}{3}\right)^2+\left(y+\frac{1}{3}\right)^2=\left(\frac{2\sqrt{2}}{3}\right)^2$, it is a circle centred at $z=-\frac{4}{3}-\frac{1}{3}i$ and radius $\frac{2\sqrt{2}}{3}$.

An ellipse: e.g. $\{z : |z+i|+|z-i|=4\}$, foci at $z=-i$ and $z=i$.

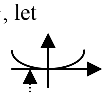




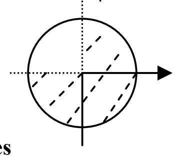
A hyperbola: e.g. $\{z : \text{Re}\, z\times \text{Im}\, z = 1\}$, above left.
e.g. $\{z : |z+5|-|z-5|=8\}$, let $z=x+yi$ to obtain $\frac{x^2}{16}-\frac{y^2}{9}=1$.



A parabola: e.g. $\{z : |\text{Im}\, z + i| = |z-i|\}$, let $z=x+yi$ to obtain $y=\frac{1}{4}x^2$.



Example- Sketch (shaded) $\{z : |z|\le 2\}\cap\left\{z : Arg z<\frac{\pi}{2}\right\}$.



Derivatives and antiderivatives involving inverse trig. functions:

$f(x)=\int F(x)dx$	$F(x)=\frac{d}{dx}f(x)$
$Sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$Cos^{-1}\frac{x}{a}$	$\frac{-1}{\sqrt{a^2-x^2}}$
$Tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$
$Sin^{-1}ax$	$\frac{a}{\sqrt{1-(ax)^2}}$
$Cos^{-1}ax$	$\frac{-a}{\sqrt{1-(ax)^2}}$
$Tan^{-1}ax$	$\frac{a}{1+(ax)^2}$

Antidifferentiation of $\frac{1}{x}$: $\int \frac{1}{x}dx = \log_e|x|+c$,

$\int \frac{a}{x\pm b}dx = a\log_e|x\pm b|+c$,

$\int\frac{1}{k(x\pm b)}dx=\frac{1}{k}\log_e|x\pm b|+c$,

$\int\frac{1}{kx\pm b}dx=\frac{1}{k}\log_e|kx\pm b|+c$,

e.g. $\int\frac{-2}{3x+1}dx=\frac{-2}{3}\log_e|3x+1|+c$.

Antidifferentiation techniques:

1) *For* $\int f(g(x))\times g'(x)dx$, *let* $u=g(x)$, e.g.

$\int 2x\sqrt{1-3x^2}\,dx$

Let $u=1-3x^2$, $\frac{du}{dx}=-6x$, $\therefore -\frac{1}{3}\frac{du}{dx}dx = 2xdx$.

$=\int -\frac{1}{3}\sqrt{u}\,du$

$=\int -\frac{1}{3}u^{\frac{1}{2}}du = -\frac{2}{9}u^{\frac{3}{2}}+c = -\frac{2}{9}(1-3x^2)^{\frac{3}{2}}+c$.

e.g. $\int\cos^2 x\sin^3 x\,dx = \int \cos^2 x(1-\cos^2 x)\sin x\,dx$

$=\int -u^2(1-u^2)du$

Let $u=\cos x$, $\frac{du}{dx}=-\sin x$, $\therefore -\frac{du}{dx}dx=\sin x\,dx$.

$=\int(-u^2+u^4)du$

$=-\frac{u^3}{3}+\frac{u^5}{5}+c=-\frac{\cos^3 x}{3}+\frac{\cos^5 x}{5}+c$.

e.g. $\int\frac{x}{\sqrt{1-x}}dx$

Let $u=1-x$, $\frac{du}{dx}=-1$, $\therefore x=1-u$, $-\frac{du}{dx}dx=dx$.

$=\int-\frac{1-u}{\sqrt{u}}du$

$=\int\left(-u^{-\frac{1}{2}}+u^{\frac{1}{2}}\right)du=-2u^{\frac{1}{2}}+\frac{2}{3}u^{\frac{3}{2}}+c$

$=-2\sqrt{1-x}+\frac{2}{3}(1-x)^{\frac{3}{2}}+c$.

2) *Using trig. identities*: e.g. $\int\sin^2 3x\,dx$

$=\int\frac{1}{2}(1-\cos 2(3x))dx=\frac{1}{2}\left(x-\frac{1}{6}\sin 6x\right)+c$.

3) *Using partial fractions*:

e.g. $\int\frac{x^2-2x-2}{x^2-3x-4}dx=\int 1+\frac{\frac{6}{5}}{x-4}-\frac{\frac{1}{5}}{x+1}dx$

$=x+\frac{6}{5}\log_e|x-4|-\frac{1}{5}\log_e|x+1|+c$

$=x+\frac{1}{5}\left(\log_e|x-4|^6-\log_e|x+1|\right)+c$

$=x+\frac{1}{5}\log_e\frac{|x-4|^6}{|x+1|}+c$.

Definite integrals: e.g. $\int_0^{\frac{\pi}{2}}(1-\sin x)\cos x\,dx$

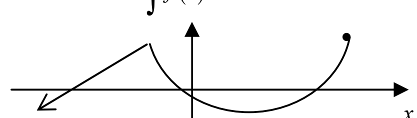
$=\int_0^1(1-u)du$

Let $u=\sin x$, $\frac{du}{dx}=\cos x$, $\frac{du}{dx}dx=\cos x\,dx$, when $x=0$, $u=0$, and when $x=\frac{\pi}{2}, u=1$.

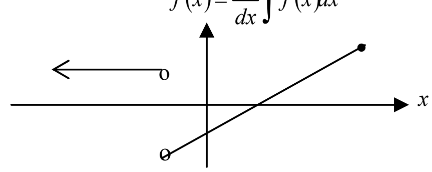
$=\left[u-\frac{1}{2}u^2\right]_0^1$

$=\frac{1}{2}$

The gradient function graph of an antiderivative of a function is the graph of the function: $\int f(x)dx$



$f(x)=\frac{d}{dx}\int f(x)dx$

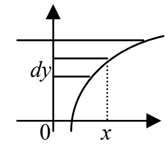


Area bounded by curves and axes:

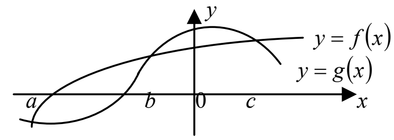
e.g. area bounded by $x=0, y=0, y=1$ and $y=\log_e x$.

$A=\int_0^1 x\,dy=\int_0^1 e^y dy$

$=\left[e^y\right]_0^1=e-1$



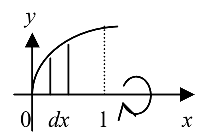
e.g. area bounded by two curves *f* and *g*. Firstly find the *x*-coordinates of the intersections by solving simultaneous eqs.

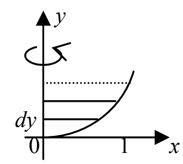


$A=\int_a^b f(x)-g(x)dx+\int_b^c g(x)-f(x)dx$

Volume of solid of revolution of a region:

e.g. 1) $y=\sqrt{x}$ e.g. 2) $y=\frac{1}{3}x^3$





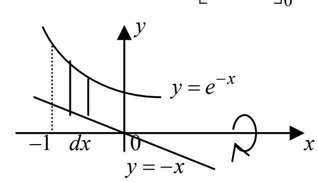
1) $V=\int_0^1\pi y^2dx=\int_0^1\pi\left(\sqrt{x}\right)^2dx=\int_0^1\pi x\,dx$

$=\left[\pi\frac{x^2}{2}\right]_0^1=\frac{\pi}{2}$.

2) When $x=1, y=\frac{1}{3}$. $x=(3y)^{\frac{1}{3}}$. $V=$

$\int_0^{\frac{1}{3}}\pi x^2dy=\int_0^{\frac{1}{3}}\pi(3y)^{\frac{2}{3}}dy=\left[\pi\frac{3^{\frac{5}{3}}y^{\frac{5}{3}}}{5}\right]_0^{\frac{1}{3}}=\frac{\pi}{5}$.

e.g.



Volume of solid of rev. of region bounded by $y=e^{-x}$, $y=-x$, $x=-1$ and y-axis

$=\int_{-1}^0\pi\left(\left(e^{-x}\right)^2-(-x)^2\right)dx$

$=\int_{-1}^0\pi\left(e^{-2x}-x^2\right)dx=\left[\pi\left(\frac{e^{-2x}}{-2}-\frac{x^3}{3}\right)\right]_{-1}^0$

$=\pi\left(\frac{1}{-2}\right)-\pi\left(\frac{e^2}{-2}+\frac{1}{3}\right)=\left(\frac{e^2}{2}-\frac{5}{6}\right)\pi$.

Second derivatives:

$f'(a)$	$f''(a)$	Nature of point at $x=a$
0	<0	Local maximum
0	>0	Local minimum
0	0	Stationary inflection pt
$\neq 0$	0	Inflection point

Implicit differentiation:

e.g. $x^2+y^2=9$, $\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}(9)$,

$2x+\frac{d}{dy}\left(y^2\right)\frac{dy}{dx}=0$, $2x+2y\frac{dy}{dx}=0$,

$\therefore\frac{dy}{dx}=-\frac{x}{y}=\pm\frac{x}{\sqrt{9-x^2}}$.

e.g. $3xy^2 = x + y$, $\frac{d}{dx}\left(3xy^2\right) = \frac{d}{dx}(x+y)$,

$3x\frac{d}{dx}\left(y^2\right) + 3y^2 = 1 + \frac{dy}{dx}$,

$6xy\frac{dy}{dx} + 3y^2 = 1 + \frac{dy}{dx}$, $6xy\frac{dy}{dx} - \frac{dy}{dx} = 1 - 3y^2$,

$(6xy-1)\frac{dy}{dx} = 1 - 3y^2$, $\therefore \frac{dy}{dx} = \frac{1-3y^2}{6xy-1}$.

Related rates: e.g. when a spherical balloon is inflated, the rate of change of its volume V is related to the rate of change of its radius r with respect to time t, i.e. $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ where $\frac{dV}{dr} = 4\pi r^2$ is obtained from $V = \frac{4}{3}\pi r^3$. It is also related to the rate of change of its surface area A with respect to time, i.e. $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$ where $A = 4\pi r^2$ and $\frac{dV}{dA} = \frac{dV}{dr} \Big/ \frac{dA}{dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$.

e.g. water is draining from a cone-shaped funnel that has height $50cm$ and radius $10cm$, the rates of change of the volume V and the depth of water h in it are related according to $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$. The relationship between r and h: $\frac{r}{h} = \frac{10}{50}$, $\therefore r = \frac{h}{5}$.

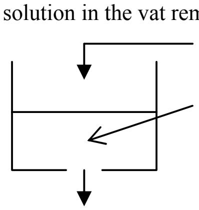
The relationship between V and h: volume of water in the cone $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{75}h^3$, $\frac{dV}{dh} = \frac{\pi}{25}h^2$, hence $\frac{dV}{dt} = \frac{\pi}{25}h^2\frac{dh}{dt}$. If $\frac{dV}{dt}$ is a constant of $-400cm^3\min^{-1}$ (negative for draining), then $\frac{dh}{dt} = -\frac{10000}{\pi h^2}$.

Setting up differential equations:

e.g. growth (decay) rate is directly proportional to population, let N be the population and $\frac{dN}{dt}$ is the growth (decay) rate, $\frac{dN}{dt} \propto N$, hence $\frac{dN}{dt} = kN$, k is the constant of proportionality, $k > 0$ for growth and $k < 0$ for decay.

e.g. rate of cooling (warming) of an object is directly proportional to the difference between the object temperature (T) and surrounding temp (T_0), $\frac{dT}{dt} = k(T - T_0)$, $k < 0$ cooling, $k > 0$ warming.

e.g. mixing solutions: Brine with a concentration of 0.2 kg of salt per litre is run into a vat initially containing 2000L of pure water at a rate of 20L per minute. The mixture is stirred continuously and allowed to run out at the same rate. Set up the differential equation for the amount of salt Q kg in the vat at time t minutes. In this case the volume of solution in the vat remains constant, 2000L.



Step 1. Rate of inflow of salt $= 0.2 \times 20 = 4kg\min^{-1}$

Step 2. At time t there is Q kg of salt in 2000L soln. $\therefore Conc = \frac{Q}{2000}kgL^{-1}$.

Step 3. Rate of outflow of salt $= \frac{Q}{2000} \times 20 = \frac{Q}{100}kg\min^{-1}$.

Rate of change in amount of salt = Rate of inflow – rate of outflow. $\therefore \frac{dQ}{dt} = 4 - \frac{Q}{100}$.

If the solution is allowed to run out at 15L per minute, the volume of solution in the vat increases at a rate of 5L per minute. At time t the volume is $(2000 + 5t)$L. Step 2. At time t there is Q kg of salt in $(2000 + 5t)$L.

$\therefore Conc = \frac{Q}{2000 + 5t}kgL^{-1}$.

Step 3. Rate of outflow of salt $= \frac{Q}{2000 + 5t} \times 15 = \frac{3Q}{400 + t}kg\min^{-1}$, $\therefore \frac{dQ}{dt} = 4 - \frac{3Q}{400 + t}$.

Methods in solving differential equations:

1) If $\frac{dy}{dx} = f(x)$, then $y = \int f(x)dx$,

e.g. $\frac{dy}{dx} = -e^{2x}$, $y = \int -e^{2x}dx$, $y = -\frac{e^{2x}}{2} + c$.

2) If $\frac{d^2y}{dx^2} = f(x)$, then $\frac{dy}{dx} = \int f(x)dx = F(x) + c$ and then $y = \int (F(x) + c)dx = \int F(x)dx + cx + k$,

e.g. $\frac{d^2y}{dt^2} = -\cos 2t$, $\frac{dy}{dt} = \int -\cos 2t dt$,

$\frac{dy}{dt} = -\frac{\sin 2t}{2} + c$, $y = \int\left(-\frac{\sin 2t}{2} + c\right)dt$,

$y = \frac{\cos 2t}{4} + ct + k$.

3) If $\frac{dy}{dx} = f(y)$, then $\frac{dx}{dy} = \frac{1}{f(y)}$, $x = \int \frac{1}{f(y)}dy$, $x = F(y) + c$, express y in terms of x if possible.

e.g. $\frac{dy}{dx} = \sqrt{4 - y^2}$, $\frac{dx}{dy} = \frac{1}{\sqrt{4 - y^2}}$,

$x = \int \frac{1}{\sqrt{2^2 - y^2}}dy$, $x = Sin^{-1}\frac{y}{2} + c$,

$y = 2Sin(x - c)$.

Numerical solution of $\frac{dy}{dx} = f(x)$ when $x = b$ using calculator, given $y = k$ when $x = a$

Use calculator to evaluate $\int_a^b f(x)dx$, then $y = \int_a^b f(x)dx + k$, e.g. given $\frac{dy}{dx} = \log_e x$ and $y = 5$ when $x = 2$, find y when $x = 3$.

$y = \int_2^3 \log_e xdx + 5 = 0.91 + 5 = 5.91$.

Numerical solution of d.e. by Euler's method:

e.g. with a step size of 0.2 solve $\frac{dy}{dx} = \cos\left(\frac{x}{2}\right)$ when $x = 0.4$, with initial condition $y = 2$ when $x = 0$. Use $y \approx y_o + \Delta x\frac{dy}{dx}$ where (x_o, y_o) is the old point, $\frac{dy}{dx}$ is the gradient at the old point and Δx is the step size. $x = 0$, $y = 2$, $\frac{dy}{dx} = \cos 0 = 1$

$x = 0.2$, $y \approx 2 + 0.2 \times 1 = 2.2$, $\frac{dy}{dx} = 0.9950$

$x = 0.4$, $y \approx 2.2 + 0.2 \times 0.9950 = 2.3990$.

Motion in a straight line: x is position from the origin at time t, Δx is change in position (or displacement) and l is distance travelled whilst Δt is time taken.

Av. velocity $v_{av} = \frac{\Delta x}{\Delta t}$, av. speed $= \frac{l}{\Delta t}$.

Velocity $v = \frac{dx}{dt}$, and speed $= |v|$. Δv is change in velocity. Av. acceleration $a_{av} = \frac{\Delta v}{\Delta t}$, acceleration $a = \frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$.

Other forms for acceleration:

$a = v\frac{dv}{dx}$, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

Given $a(t)$, $v(t) = \int a(t)dt$, $x(t) = \int v(t)dt$.

Given $a(x)$, $\frac{1}{2}v^2 = \int a(x)dx$ to give $v(x)$, let $\frac{dx}{dt} = v(x)$, $\frac{dt}{dx} = \frac{1}{v(x)}$, $t = \int \frac{1}{v(x)}dx$, then express x in terms of t to give $x(t)$.

Given $a(v)$, $v\frac{dv}{dx} = a(v)$, $\frac{dv}{dx} = \frac{a(v)}{v}$, $\frac{dx}{dv} = \frac{v}{a(v)}$, $x = \int \frac{v}{a(v)}dv$, then express v in terms of x to give $v(x)$ etc.

Given $a = vf(x)$, $v\frac{dv}{dx} = vf(x)$, $\frac{dv}{dx} = f(x)$, $v = \int f(x)dx$ to give $v(x)$ etc.

e.g. Find $a(x)$, $x(t)$ and $a(t)$, given $v = 1 - x$ and at $t = 0$, $x = 0$. $\frac{1}{2}v^2 = \frac{1}{2}(1-x)^2$,

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}(1-x)^2\right) = -(1-x)$.

$v = 1 - x$, $\frac{dx}{dt} = 1 - x$, $\frac{dt}{dx} = \frac{1}{1-x}$,

$t = \int \frac{1}{1-x}dx = -\log_e|1-x| + c$. At $t = 0$, $x = 0$, $\therefore c = 0$, $\therefore t = -\log_e|1-x|$, hence $x = 1 \pm e^{-t}$ and only $x = 1 - e^{-t}$ satisfies the initial conditions. $a = -(1-x) = -e^{-t}$.

e.g. Find $v(x)$ and then $x(t)$, $v(t)$ and $a(t)$, given $a = -x$ and at $t = 0$, $x = 0$, $v = 1$.

$a = -x$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -x$, $\frac{1}{2}v^2 = \int -xdx$,

$\frac{1}{2}v^2 = -\frac{1}{2}x^2 + c$, $x = 0$, $v = 1 \rightarrow c = \frac{1}{2}$,

$v^2 = 1 - x^2$, $v = \pm\sqrt{1-x^2}$. $\frac{dx}{dt} = \pm\sqrt{1-x^2}$,

$t = \int \frac{\pm dx}{\sqrt{1-x^2}} = Sin^{-1}x + c$ or $Cos^{-1}x + d$,

$t = 0$, $x = 0 \rightarrow c = 0$, $d = \frac{\pi}{2}$, hence

$x = Sin\,t$ or $x = Cos\left(t - \frac{\pi}{2}\right)$, they are the same result. Drop the restrictions on t, choose $x = \sin t$, $v = \sqrt{1 - \sin^2 t} = \cos t$ to satisfy $t = 0$, $v = 1$. $a = -x = -\sin t$.

Velocity-time graphs of motion in a straight line:

Gradient of tangent $\frac{dv}{dt}$ = acceleration a.

Area bounded by curve and time axis $\int_{t_1}^{t_2} v\,dt = \Delta x$,

i.e. change in position (or displacement).

Equations of motion in a straight line under constant acceleration ***a*****:**

s represents displacement Δx , not x, in the following equations. u is the velocity at $t = 0$, i.e. the initial velocity, v is the velocity at time t.

1) $v = u + at$, 2) $s = \frac{1}{2}(u + v)t$,

3) $s = ut + \frac{1}{2}at^2$, 4) $s = vt - \frac{1}{2}at^2$,

5) $v^2 = u^2 + 2as$. Each equation contains four of the five quantities. Select the appropriate equation to suit the given information.

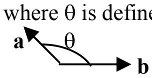
Linearly dependent vectors: Given a number of vectors, they are linearly dependent if any one of them can be expressed as a linear combination of all the other vectors, e.g. $\mathbf{x} = 2\mathbf{y} - \mathbf{z}$, then **x**, **y** and **z** are linearly dependent. Note that $\mathbf{y} = ½\mathbf{x} + ½\mathbf{z}$ and $\mathbf{z} = -\mathbf{x} + 2\mathbf{y}$. Also $\mathbf{x} - 2\mathbf{y} + \mathbf{z} = \mathbf{0}$.

In general, if there are non-zero scalars p, q and r such that $p\mathbf{x} + q\mathbf{y} + r\mathbf{z} = \mathbf{0}$, then **x**, **y** and **z** are linearly dependent. Otherwise they are **linearly independent**, i.e. if $p\mathbf{x} + q\mathbf{y} + r\mathbf{z} = \mathbf{0}$ only when p = q = r = 0, then **x**, **y** and **z** are linearly independent, e.g. three non-coplanar vectors are linearly independent because none can be expressed as a linear combination of the other two without having non-zero coefficients.

Unit vector: Unit vector in the same direction as $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is

$$\hat{r} = \frac{1}{\sqrt{a^2 + b^2 + c^2}}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}).$$

Scalar product: $\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$, where θ is defined as the angle between **a** and **b** when they are placed tail to tail.



Parallel vectors: Two vectors **a**, **b** are parallel if one is a scalar multiple of the other, i.e. $\mathbf{b} = n\mathbf{a}$. Two parallel vectors are linearly dependent because $\mathbf{b} - n\mathbf{a} = \mathbf{0}$. The angle between two parallel vectors is 0° or 180°. $\mathbf{a} \cdot \mathbf{b} = \pm ab$.

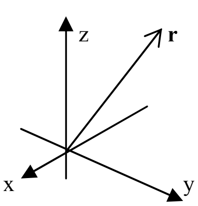
Perpendicular vectors: Two non-zero vectors **a**, **b** are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$. Two perpendicular vectors are linearly independent.

Angle between a vector and x, y or z-axis: $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ makes angles α, β, γ with x, y, z-axis respectively, where

$$\alpha = Cos^{-1}\frac{a}{\sqrt{a^2 + b^2 + c^2}},$$

$$\beta = Cos^{-1}\frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

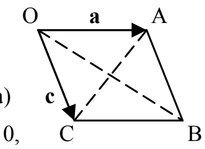
$$\gamma = Cos^{-1}\frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$



Scalar and vector resolutes of vector q parallel and perpendicular to vector p:

1. Find unit vector **ŝ** parallel to **s**.
2. Scalar resolute of **q** parallel to **s** = **q . ŝ**
3. Vector resolute of **q** parallel to **s** = (**q . ŝ**) **ŝ**
4. Vector resolute of **q** ⊥ to **s** = **q** – (**q . ŝ**) **ŝ**
5. Scalar resolute of **q** ⊥ to **s** = |**q** – (**q . ŝ**) **ŝ**|

Vector proofs: e.g. the diagonals of a rhombus are perpendicular. Proof: Let OABC be a rhombus, $\overline{OA} = \overline{AB} = \overline{BC} = \overline{OC}$. Introduce $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{c} = \overrightarrow{OC}$.



$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$, $\overrightarrow{OB} = \mathbf{c} + \mathbf{a}$,

$\overrightarrow{AC} \cdot \overrightarrow{OB} = (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} + \mathbf{a})$

$= \mathbf{c} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{a} = c^2 - a^2 = 0$,

since OABC is a rhombus, $c = a$.

$\therefore \overrightarrow{AC}$ and $\overrightarrow{OB}$ are perpendicular.

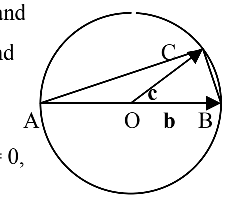
e.g. the angle subtended by a diameter in a circle is a right angle. Proof: $\overline{AB}$ is a diameter. O is the centre. Introduce $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. $\therefore \overrightarrow{AC} = \mathbf{c} + \mathbf{b}$ and $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$. $\overrightarrow{AC} \cdot \overrightarrow{BC}$

$= (\mathbf{c} + \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b})$

$= \mathbf{c} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{b} = c^2 - b^2 = 0$,

since $b = c$ = radius.

$\therefore \angle ACB$ is a right angle.



2-D vector equation, parametric equations, cartesian equation and graph of locus:

e.g. Let **r** be a position vector,

vector equation $\mathbf{r}(t) = t^2\mathbf{i} - (t-1)\mathbf{j}$, $t \ge 0$.

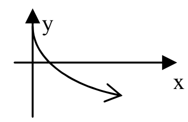
Parametric equations are $x = t^2$, $y = -(t-1)$.

$\therefore t = \sqrt{x}$, $\therefore y = -\sqrt{x} + 1$

is the cartesian eqn.

At $t = 0$, $x = 0$ and $y = 1$,

$(0,1)$ is the starting point.



Vector calculus: e.g. refer to the above **r**(t),

$\mathbf{v} = \frac{d}{dt}\mathbf{r}(t) = 2t\mathbf{i} - \mathbf{j}$, $\mathbf{a} = \frac{d}{dt}\mathbf{v}(t) = 2\mathbf{i}$,

speed v = |**v**| = $\sqrt{(2t)^2 + (-1)^2} = \sqrt{4t^2 + 1}$.

Particle has a constant acceleration 2**i**, initial velocity **v**(0) = –**j**, initial speed v(0) = 1.

Average velocity between $t = 1$ and $t = 3$:

$\mathbf{r}(1) = \mathbf{i}$, $\mathbf{r}(3) = 9\mathbf{i} - 2\mathbf{j}$, $\mathbf{v}_{av} = \frac{(9i - 2j) - i}{3 - 1} = 4\mathbf{i} - \mathbf{j}$.

Velocity vector **v**(t) is always tangential to the locus (path).

e.g. a particle with position vector **r**(t), $t \ge 0$,

$\mathbf{r}(t) = 3\cos 2t\ \mathbf{i} + 3\sin 2t\ \mathbf{j}$.

$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = -6\sin 2t\ \mathbf{i} + 6\cos 2t\ \mathbf{j}$.

$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = -12\cos 2t\ \mathbf{i} - 12\sin 2t\ \mathbf{j}$.

Note that 1) $\mathbf{a}(t) = -4\mathbf{r}(t)$, **a**($t$) and **r**($t$) are parallel and opposite in direction. 2) $\mathbf{v}(t) \cdot \mathbf{r}(t) = 0$, **v**($t$) and **r**($t$) are perpendicular. 3) Hence **a**(t) and **v**(t) are also perpendicular. 4) $|\mathbf{r}(t)| = 3$ implying the locus is a circle centred at the origin. This can also be shown by finding the cartesian equation to be $x^2 + y^2 = 9$. $\left[x = 3\cos 2t, y = 3\sin 2t\right.$, $\left.x^2 + y^2 = (3\cos 2t)^2 + (3\sin 2t)^2 = 9\right]$.

5) Speed v=|**v**(t)|= $\sqrt{(-6\sin 2t)^2 + (6\cos 2t)^2} = 6$, it is constant.

6) Magnitude of **a** = |**a**(t)| = 12 is also constant.

e.g. Given $\mathbf{a} = 5\mathbf{i}$, when $t = 0$, $\mathbf{r} = 0$, $\mathbf{v} = 7\mathbf{i} + 10\mathbf{j}$.

Find **v**(t) and **r**(t). $\mathbf{v}(t) = \int a\,dt = \int 5\ \mathbf{i}\ dt = 5t\mathbf{i} + \mathbf{c}$,

when $t = 0$, $\mathbf{v} = 7\mathbf{i} + 10\mathbf{j}$, $\therefore \mathbf{c} = 7\mathbf{i} + 10\mathbf{j}$ and $\mathbf{v}(t) = (5t + 7)\mathbf{i} + 10\mathbf{j}$.

$\mathbf{r}(t) = \int v(t)dt = \int (5t + 7)\mathbf{i} + 10\mathbf{j}\ dt$

$= \left(\frac{5}{2}t^2 + 7t\right)\mathbf{i} + 10t\mathbf{j} + \mathbf{d}$, when $t = 0$, $\mathbf{r} = 0$,

$\therefore$ **d = 0** and $\mathbf{r}(t) = \left(\frac{5}{2}t^2 + 7t\right)\mathbf{i} + 10t\mathbf{j}$.

The particle has constant acceleration in the **i** direction, thus the **i** component of **v** changes with time and the **j** component remains at 10. The path of the particle:

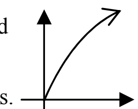
$x = \left(\frac{5}{2}t^2 + 7t\right)$, $y = 10t$,

$\therefore x = \frac{y^2}{40} + \frac{7y}{10}$, $y^2 + 28y - 40x = 0$,

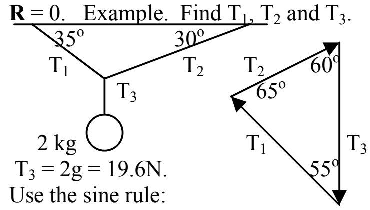
$\therefore y = \pm 2\sqrt{10x + 49} - 14$ and

only $y = 2\sqrt{10x + 49} - 14$

satisfies the initial conditions.



Mechanics: Weight is the force of gravity on particle, **w** = m**g**. **Momentum** is defined as the product of mass and velocity, **p** = m**v**. **Newton's third law**, for every **action** force exerted by particle A on particle B, there is a **reaction** force of the same magnitude exerted by B on A in the opposite direction. **Friction** is a force between two surfaces that slide or tend to slide over each other. The direction of friction force is always opposite to the direction of motion of particle. If the particle is at rest relative to the other surface, 0 ≤ friction < μN, where N is the normal reaction on the particle. If it is sliding or on the verge of sliding, friction = μN. **Resultant force R** is the **vector sum** of all the forces acting on a particle. Forces on other particles do not contribute to the resultant force. **Equation of motion** is established by means of **Newton's second law**, **a** = **R**/m. In using this equation, **a** is in ms^{-2}, **R** in newtons (N), m in kg. When a particle is in **equilibrium**, **R** = 0. Example. Find T_1, T_2 and T_3.



$T_3 = 2g = 19.6$N.

Use the sine rule:

$\frac{T_1}{\sin 60^o} = \frac{T_2}{\sin 55^o} = \frac{T_3}{\sin 65^o}$ etc.

Example. A 5 kg particle slides down a plane inclined at 30° to the horizontal and μ = 0.2.

Weight is resolved into two **i** and **j** components.

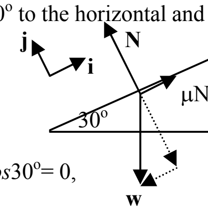
Resultant force:

j-comp, $^+$N + $^-$5g*cos*30°= 0,

$\therefore$ N = 5g*cos*30°.

i-comp, $^-$5g*sin*30° + $^+$μN = 5a,

$\therefore$ a = $g\left(\frac{\sqrt{3} - 5}{10}\right)$, it is a negative value,



i.e. it is opposite to **i**.

Connected particles: Each particle has its own equation of motion. The magnitude of acceleration is the same for all.

Check for accuracy before use.
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