

Section 1

1	2	3	4	5	6	7	8	9	10	11
C	E	D	A	C	A	C	A	A	B	E

12	13	14	15	16	17	18	19	20	21	22
C	A	C	A	A	D	A	B	E	B	C

Q1 The maximum value of $(\sin^{-1} x)^2 - (\sin x)^{-2}$ is $(\sin^{-1}(1))^2 - (\sin(1))^{-2} \approx 1.316$. C

Q2 $\cos(x - y + z) = \cos(x - (y - z))$
 $= \cos(x)\cos(y - z) + \sin(x)\sin(y - z)$. E

Q3 Equation of the inverse is $x = 2 \arctan\left[\frac{1}{2}(y+1)\right] - 1$.

$$\frac{x+1}{2} = \arctan\left[\frac{1}{2}(y+1)\right], \frac{1}{2}(y+1) = \tan\left(\frac{x+1}{2}\right),$$

$$y = 2 \tan\left(\frac{x+1}{2}\right) - 1, \text{ where } -\frac{\pi}{2} < \frac{x+1}{2} < \frac{\pi}{2}, \text{ i.e. } -\pi - 1 < x < \pi - 1. \text{ D}$$

Q4 $y = \left(x - a + \frac{1}{\sqrt{x-a}}\right)\left(x - a - \frac{1}{\sqrt{x-a}}\right) + b$
 $y = (x-a)^2 - \frac{1}{x-a} + b$. Only one vertical asymptote $x = a$. A

Q5
 $\frac{(x+1)^2}{4} + (y-2)^2 = 1 \rightarrow \frac{x^2}{4} + (y-2)^2 = 1 \rightarrow \frac{x^2}{4} + (y-1)^2 = 1$
 $\rightarrow \frac{x^2}{4} + \left(\frac{y}{2} - 1\right)^2 = 1, \text{ i.e. } x^2 + (y-2)^2 = 4$. C

Q6 $z = i\left(\cos\left(\frac{\pi}{2} + \theta\right) - i\sin\left(\frac{\pi}{2} - \theta\right)\right) = i(-\sin\theta - i\cos\theta)$
 $= \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$. $\therefore \text{Arg}(z) = -\theta$. A

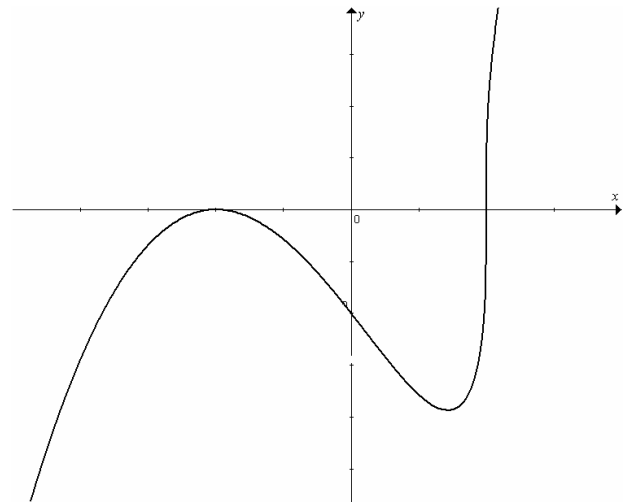
Q7 $2z^2 - (5-i)z + 3 + 11i = 2(z - (3-2i))(z - (a+bi))$
 $= 2z^2 - ((2a+6) + (2b-4)i)z + (6a+4b) + (6b-4a)i$
 $\therefore 2a+6=5 \text{ and } 2b-4=-1, \therefore a=-\frac{1}{2} \text{ and } b=\frac{3}{2}$.

The other root is $-\frac{1}{2} + \frac{3}{2}i$. C

Q8 $3\text{cis}\left(\frac{\pi}{3}\right) - i\text{cis}\left(-\frac{\pi}{6}\right)$
 $= 3\cos\left(\frac{\pi}{3}\right) + i3\sin\left(\frac{\pi}{3}\right) - i\cos\left(-\frac{\pi}{6}\right) + \sin\left(-\frac{\pi}{6}\right)$
 $= \frac{3}{2} + \frac{3\sqrt{3}}{2}i - \frac{\sqrt{3}}{2}i + -\frac{1}{2} = 1 + i\sqrt{3} = 2\text{cis}\left(\frac{\pi}{3}\right)$. A

Q9 A

Q10 Choose appropriate values of $a, b \in \mathbb{R}$, use graphics calculator to draw the graph of $y = 1000(x+a)^2(x-b)^{\frac{1}{3}}$. B



Q11 $\frac{d}{dx}\left(\frac{-1}{\sqrt{a-x^2}}\right) = \frac{d}{dx}\left(-\left(a-x^2\right)^{-\frac{1}{2}}\right)$
 $= \frac{1}{2}\left(a-x^2\right)^{-\frac{3}{2}}(-2x) = -\frac{x}{\left(a-x^2\right)\sqrt{a-x^2}}$. E

Q12 $2x^2y^2 = x^2 - y^2, 4xy^2 + 2x^2 \cdot 2y \frac{dy}{dx} = 2x - 2y \frac{dy}{dx}$,
 $\therefore \frac{dy}{dx} = \frac{2x - 4xy^2}{2y + 4x^2y} = \frac{x - 2xy^2}{y + 2x^2y}$.

At $x=1, 2(1)^2 y^2 = (1)^2 - y^2, 3y^2 = 1, y = \pm \frac{1}{\sqrt{3}}$,

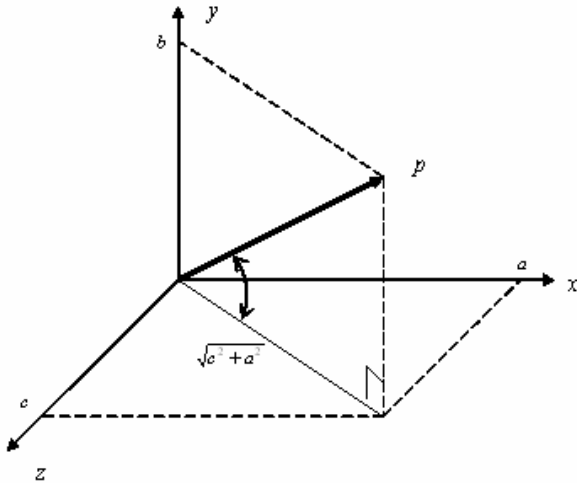
and $\frac{dy}{dx} = \frac{1 - \frac{2}{3}}{\pm \sqrt{3}} = \pm \frac{1}{3\sqrt{3}}$. C

Q13 Since $x \in \mathbb{R} \setminus [-1, 1], \therefore x^2 > 1$,
 $\therefore \int \frac{1}{1-x^2} d(x^2) = -\log_e |1-x^2| + c = -\log_e (x^2 - 1) + c$. A

Q14 $y' = y(a-y)$, when $y=2$,
 gradient of slope (from graph) $= y' \approx \frac{2.0}{0.5} = 4$,
 $\therefore 4 \approx 2(a-2), \therefore a \approx 4$. C

Q15 $y = \int_{-\frac{1}{2}}^{-1} \tan(x^2) dx - 1$. Use graphics calculator to evaluate the definite integral, $y \approx -0.356 - 1 = -1.356$. A

Q16 A



Q17 $c - 3a + 2b = 0$, $\therefore a, b$ and c are dependent. D

Q18 The direction of motion is given by the unit vector in the direction of the velocity vector.

$$\mathbf{r}(t) = \tan(t)\mathbf{i} + \frac{1}{2}\sec^2(t)\mathbf{j} + \mathbf{k}, \quad \mathbf{r}'(t) = \tan(t)\mathbf{i} + \frac{1}{2}(1 + \tan^2(t))\mathbf{j} + \mathbf{k}$$

$$\begin{aligned} \mathbf{u}(t) &= \frac{d}{dt}\mathbf{r}'(t) = \sec^2(t)\mathbf{i} + \tan(t)\sec^2(t)\mathbf{j} \\ &= (1 + \tan^2(t))\mathbf{i} + \tan(t)(1 + \tan^2(t))\mathbf{j}. \end{aligned}$$

When $t = \frac{3\pi}{4}$, $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j}$.

$$\therefore \hat{\mathbf{u}} = \frac{1}{2\sqrt{2}}(2\mathbf{i} - 2\mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}. \quad \text{A}$$

Q19 Initial velocity $= \frac{1}{0.20}(3\mathbf{i} - \mathbf{j} + \mathbf{k})$,

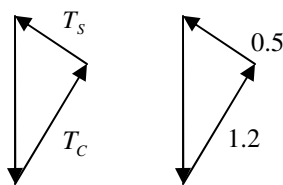
$$\therefore \text{initial speed} = \frac{1}{0.20}\sqrt{3^2 + (-1)^2 + 1^2} = 5\sqrt{11}.$$

Final velocity $= \frac{1}{0.20}(\mathbf{i} - \mathbf{k})$,

$$\therefore \text{final speed} = \frac{1}{0.20}\sqrt{1^2 + (-1)^2} = 5\sqrt{2}.$$

$$\therefore \text{change of speed} = 5\sqrt{2} - 5\sqrt{11} \approx -9.512. \quad \text{B}$$

Q20 $\sqrt{1.2^2 + 0.5^2} = 1.3$, right-angled triangle.



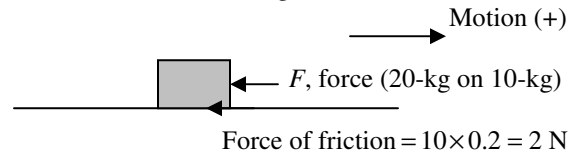
$$\frac{T_s}{T_c} = \frac{0.5}{1.2} \approx 0.42. \quad \text{E}$$

Q21 The particle moves at constant speed and direction, \therefore the resultant force on the particle is zero. Hence the reaction force of the slope on the particle is equal but opposite to the weight force on the particle. B

Q22 Consider the two boxes as a single object. Total force of friction $= 30 \times 0.2 = 6$ N opposite to motion,

$$\therefore a = \frac{6}{30} = 0.2 \text{ ms}^{-1} \text{ opposite to motion.}$$

Consider the forces on the 10-kg box:



$$R = ma, \quad -F + 2 = 10 \times (-0.2), \quad \therefore F = 0. \quad \text{C}$$

Section 2

Q1a. $a^2 = 1 - v^2$, $a = \pm\sqrt{1 - v^2}$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \pm\sqrt{1 - v^2}$.

Q1bi. $\frac{1}{2}\frac{d(v^2)}{dx} = \pm\sqrt{1 - v^2}$, $\frac{dx}{d(v^2)} = \pm\frac{1}{2\sqrt{1 - v^2}}$,

$$x = \mp \int \frac{1}{2\sqrt{1 - (v^2)}} d(v^2) = \mp \int \frac{1}{2}(1 - (v^2))^{-\frac{1}{2}} d(v^2) = \mp\sqrt{1 - v^2} + c.$$

At $x = \pm 1$, $v = 0$, $\therefore c = 0$, $\therefore x = \mp\sqrt{1 - v^2}$.

Hence $v = \pm\sqrt{1 - x^2}$. $\therefore A = \pm 1$, $B = 1$ and $C = -1$.

Q1bii. $v = \frac{dx}{dt} = \pm\sqrt{1 - x^2}$, $\frac{dt}{dx} = \pm\frac{1}{\sqrt{1 - x^2}}$. Note: both signs

give the same result.

Choose $\frac{dt}{dx} = -\frac{1}{\sqrt{1 - x^2}}$, $t = \int -\frac{1}{\sqrt{1 - x^2}} dx = \cos^{-1}(x) + c$.

When $t = 0$, $x = 1$, $\therefore c = 0$. $\therefore t = \cos^{-1}(x)$, where $0 \leq t \leq \pi$.

Hence $x = \cos(t)$. $\therefore D = 1$ and $n = 1$.

Q1c. $\int_0^\pi v dt = \int_0^\pi \frac{dx}{dt} dt = \int_1^{-1} dx = [x]_1^{-1} = -2$.

Q1d. $x = \cos(t)$, $v = -\sin(t)$, $a = -\cos(t)$, $\therefore a = -x$.

Alternatively, refer to Q1a and b,

$$a = \pm\sqrt{1 - v^2} \text{ and } x = \mp\sqrt{1 - v^2}, \quad \therefore a = -x.$$

Q2a. At $x=6$, $y=1+e$.

$$\begin{aligned} \text{Area of shaded region} &= \int_0^6 (1+e^{x-5}) dx + (1+e) \times 1 \\ &= [x + e^{x-5}]_0^6 + 1+e = 6+e - e^{-5} + 1+e = 7+2e - e^{-5} \text{ m}^2. \end{aligned}$$

Q2bi $y=1+e^{x-5}$, $x = \log_e(y-1)+5$

At $x=0$, $y=1+e^{-5}$.

Max volume of water

$$= \int_{1+e^{-5}}^{1+e} \pi x^2 dy = \int_{1+e^{-5}}^{1+e} \pi (\log_e(y-1)+5)^2 dy \text{ m}^3.$$

Q2bii Volume of concrete

$$= \pi 7^2(1+e) - \int_{1+e^{-5}}^{1+e} \pi (\log_e(y-1)+5)^2 dy$$

$\approx 350 \text{ m}^3$. Evaluate the definite integral with graphics calculator.

Q2c. Max volume = $\int_{1+e^{-5}}^{1+e} \pi (\log_e(y-1)+5)^2 dy = 221.99 \text{ m}^3$.

Time required = $\frac{221.99}{12} = 18.499 \text{ min}$.

Average rate of increase in depth of water

$$= \frac{(1+e) - (1+e^{-5})}{18.499} = 0.146576 \text{ m min}^{-1} = 2.44 \text{ mm s}^{-1}.$$

Q2di. $V(h) = \int_{1+e^{-5}}^h \pi (\log_e(y-1)+5)^2 dy$.

Q2dii. $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$,

$$\therefore \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}} = \frac{12}{\frac{d}{dh} \left(\int_{1+e^{-5}}^h \pi (\log_e(y-1)+5)^2 dy \right)}$$

Q3ai. $r(t) = \left(t+1 + \frac{1}{t+1} \right) i + \left(t+1 - \frac{1}{t+1} \right) j$, $t \geq 0$.

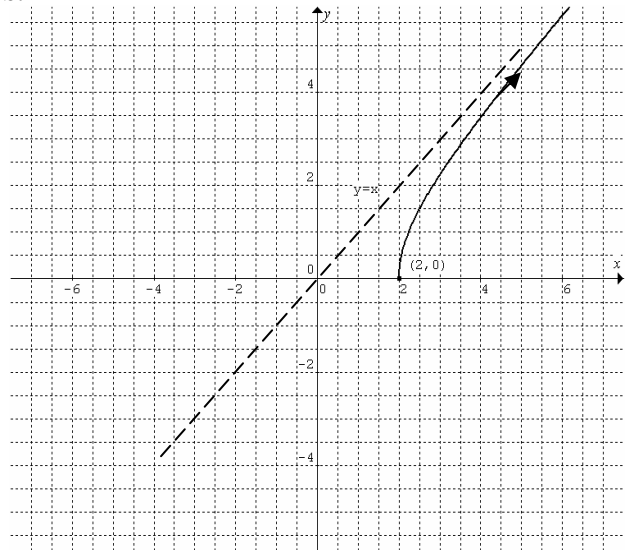
$$x = t+1 + \frac{1}{t+1}, \quad y = t+1 - \frac{1}{t+1}, \quad \therefore x+y = 2(t+1),$$

$$\therefore t+1 = \frac{x+y}{2}, \quad \therefore x = \frac{x+y}{2} + \frac{2}{x+y}, \quad x = \frac{(x+y)^2 + 4}{2(x+y)},$$

$$\therefore 2x(x+y) = (x+y)^2 + 4. \text{ Expand and simplify to } \frac{x^2}{4} - \frac{y^2}{4} = 1.$$

Since $t \geq 0$, $\therefore x \geq 2$ and $y \geq 0$.

Q3b.



Q3ci. $P(\sqrt{2}, 3)$. Let $\left(t+1 + \frac{1}{t+1}, t+1 - \frac{1}{t+1} \right)$ be the coordinates of the particle at time t .

Required vector = $\left(t+1 + \frac{1}{t+1} - \sqrt{2} \right) i + \left(t+1 - \frac{1}{t+1} - 3 \right) j$.

Q3cii. Distance $D = \sqrt{\left(t+1 + \frac{1}{t+1} - \sqrt{2} \right)^2 + \left(t+1 - \frac{1}{t+1} - 3 \right)^2}$.

Use graphics calculator to find the shortest distance.

$D_{\min} = 1.732$

Q3ciii. Time = 1.414

Q3d. $v(t) = \frac{d}{dt} r(t) = \left(1 - \frac{1}{(t+1)^2} \right) i + \left(1 + \frac{1}{(t+1)^2} \right) j$, $t \geq 0$.

At $t = \sqrt{2}$, $v = \left(1 - \frac{1}{(\sqrt{2}+1)^2} \right) i + \left(1 + \frac{1}{(\sqrt{2}+1)^2} \right) j$.

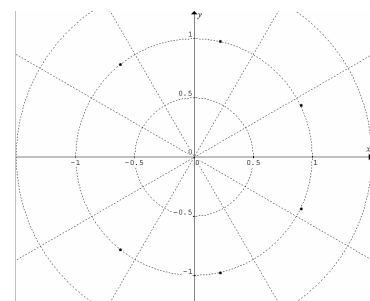
Speed = $\sqrt{0.828^2 + 1.172^2} \approx 1.4$.

Q4a. $\frac{z^7 + 1}{z + 1} = \frac{(z+1)(z^6 - z^5 + z^4 - z^3 + z^2 - z + 1)}{z + 1}$

$$= z^6 - z^5 + z^4 - z^3 + z^2 - z + 1.$$

$\therefore A = C = E = G = 1$ and $B = D = F = -1$.

Q4b. Since $z^7 + 1 = (z+1)P(z)$, the roots of $P(z) = 0$ are the roots of $z^7 + 1 = 0$ except $z = -1$.



Q4c. $z = cis\left(\pm \frac{\pi}{7}\right), cis\left(\pm \frac{3\pi}{7}\right), cis\left(\pm \frac{5\pi}{7}\right).$

Q4d. $P(z) = z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$
 $= (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6),$
 $\therefore z_1 z_2 z_3 z_4 z_5 z_6 = 1$

Q4e. The z term in the expansion of $(z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)(z - z_6)$ is $-(z_1 + z_2 + z_3 + z_4 + z_5 + z_6)z$. $\therefore z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = -1$.

$\therefore cis\left(-\frac{\pi}{7}\right) + cis\left(\frac{\pi}{7}\right) + cis\left(-\frac{3\pi}{7}\right) + cis\left(\frac{3\pi}{7}\right) + cis\left(-\frac{5\pi}{7}\right) + cis\left(\frac{5\pi}{7}\right) = -1$

$\therefore \cos\left(-\frac{\pi}{7}\right) + \cos\left(\frac{\pi}{7}\right) + \cos\left(-\frac{3\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(-\frac{5\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = -1$

$+ i\left(\sin\left(-\frac{\pi}{7}\right) + \sin\left(\frac{\pi}{7}\right) + \sin\left(-\frac{3\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) + \sin\left(-\frac{5\pi}{7}\right) + \sin\left(\frac{5\pi}{7}\right)\right) = 0$

$\therefore \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = 1$

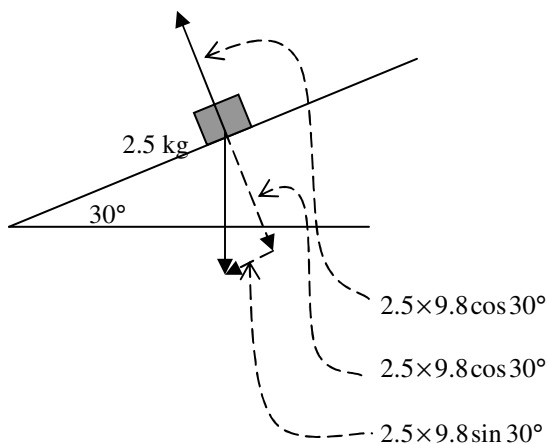
$+ i\left(-\sin\left(\frac{\pi}{7}\right) + \sin\left(\frac{\pi}{7}\right) - \sin\left(\frac{3\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) - \sin\left(\frac{5\pi}{7}\right) + \sin\left(\frac{5\pi}{7}\right)\right) = 0$

$\therefore 2\cos\left(\frac{\pi}{7}\right) + 2\cos\left(\frac{3\pi}{7}\right) + 2\cos\left(\frac{5\pi}{7}\right) = 1$

$\therefore \cos\left(\frac{\pi}{7}\right) + \cos\left(\frac{3\pi}{7}\right) + \cos\left(\frac{5\pi}{7}\right) = \frac{1}{2}$.

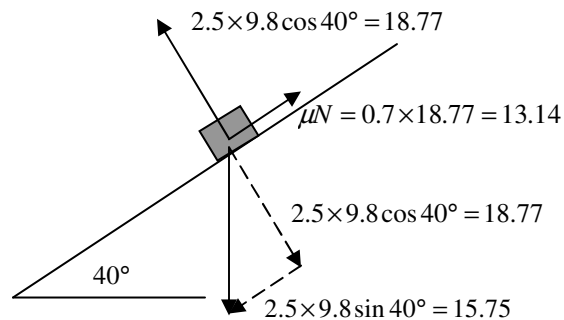
Q5a. The force of friction is zero.

Q5b.



The component of the weight force along the plane is $2.5 \times 9.8 \sin 30^\circ = 12.25 \text{ N}$ and it is less than the sliding friction $\mu N = 0.7 \times (2.5 \times 9.8 \cos 30^\circ) = 14.85 \text{ N}$.
 \therefore the object is at rest, and the friction preventing it from sliding is 12.25 N .

Q5ci.



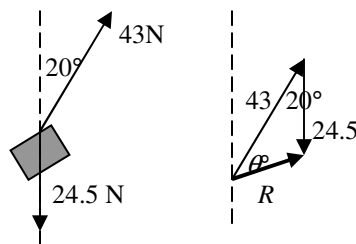
Resultant force = $+15.75 + -13.14 = +2.61 \text{ N}$,

$\therefore a = \frac{R}{m} = \frac{+2.61}{2.5} = +1.044 \approx +1.04 \text{ ms}^{-2}$.

Q5cii. $a = +1.044, u = 0, s = +3.0, v^2 = u^2 + 2as,$
 $v = 2.503 \text{ ms}^{-1}$.

|Momentum| = $mv = 2.5 \times 2.503 = 6.26 \text{ kg ms}^{-1}$.

Q5d The 43 N pulling force has a vertical component of $43 \cos 20^\circ = 40.41 \text{ N}$, and it is greater than the weight $2.5 \times 9.8 = 24.5 \text{ N}$ of the object. Hence the object takes off from the plank. The following diagram shows the two forces on the object.



$R = \sqrt{43^2 + 24.5^2 - 2(43)(24.5)\cos 20^\circ} = 21.664 \approx 21.7 \text{ N}$

Q5e. $\frac{\sin \theta^\circ}{24.5} = \frac{\sin 20^\circ}{21.664}, \theta^\circ = 22.8^\circ$.

The direction of the resultant force R is $20^\circ + 22.8^\circ \approx 43^\circ$ with the vertical, which is also the direction of motion.

Q5f. The object takes off from rest under a constant resultant force of 21.7 N at 43° with the vertical. It moves in a straight line into the air in the same direction as the resultant force and accelerates at $\frac{21.7}{25} \approx 8.7 \text{ ms}^{-2}$.

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