2003 VCAA Specialist Mathematics Exam 2 Solutions
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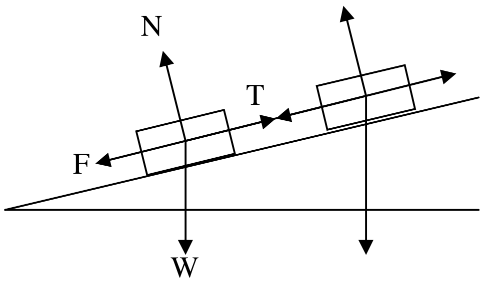
Q1a

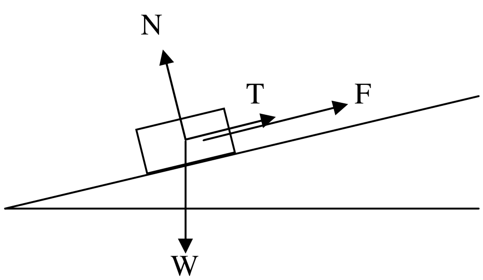


$W = 1200g$, $N = W\cos 8^o = 1200g\cos 8^o$,
$\therefore$ Friction $F = \mu N = 0.09\times 1200g\cos 8^o$.

Apply Newton's second law to sled along the slope:
$R = ma$

$^+T +^- 0.09\times 1200g\cos 8^o +^- 1200g\sin 8^o = 1200\times^+ 0.25$
$T = 2985$ newtons

Q1b



Apply Newton's first law to sled along the slope:
$R = 0$
$^+T +^+ F +^- 1200g\sin 8^o = 0$,
$T = -F + 1200g\sin 8^o$

Q1ci $T = -\mu N + 1200g\sin 8^o$
$= -0.09\times 1200g\cos 8^o + 1200g\sin 8^o$
$= 589$ newtons

Q1cii If $\mu = 0.15$, maximum friction possible is $\mu N = 1747$ newtons that is greater than $1200g\sin 8^o = 1637$ newtons. Friction alone can prevent the sled from sliding down the slope.
$\therefore T = 0$ newtons

Q2a $\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$
$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2}$
$= \frac{\sqrt{3}+1}{2\sqrt{2}}$
$= \frac{\sqrt{6}+\sqrt{2}}{4}$

$\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$
$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2}$
$= \frac{\sqrt{3}-1}{2\sqrt{2}}$
$= \frac{\sqrt{6}-\sqrt{2}}{4}$

Q2b $v - u = -\frac{\sqrt{6}-\sqrt{2}}{2}i$, $Arg(v-u) = -\frac{\pi}{2}$.

Q2c $\frac{v}{u} = \frac{cis\left(-\frac{\pi}{12}\right)}{cis\left(\frac{\pi}{12}\right)} = cis\left(-\frac{\pi}{12}-\frac{\pi}{12}\right)$
$= cis\left(-\frac{\pi}{6}\right)$
$Arg\left(\frac{v}{u}\right) = -\frac{\pi}{6}$.

Q2d Since u and v satisfy $z^2 + az + b = 0$,
$\therefore (z-u)(z-v) = 0$,
$\therefore z^2 - (u+v)z + uv = 0$.

Equate coefficients,
$b = uv = cis\left(\frac{\pi}{12}\right)cis\left(-\frac{\pi}{12}\right)$
$= cis0 = 1$
$a = -(u+v) = -\frac{\sqrt{6}+\sqrt{2}}{2}$.

Q3a $\overrightarrow{OA} = 4\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$

$\overrightarrow{OB} = 2\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{OC} = -2\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k} = \overrightarrow{OC}$

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = 4\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k} = \overrightarrow{OA}$

$\overline{OA} = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$

$\overline{OC} = \sqrt{2^2 + 5^2 + 2^2} = \sqrt{33}$

$\therefore \overline{OA} = \overline{AB} = \overline{CB} = \overline{OC}$

$\overrightarrow{OA} \bullet \overrightarrow{OC} = -8 + 5 - 8 \neq 0$,

$\therefore$ $\angle AOC$ is not a right angle.

$\therefore$ $OABC$ is a rhombus.

Q3b $\overrightarrow{OA} \bullet \overrightarrow{OC} = \overline{OA} \times \overline{OC} \cos\theta$

$-8 + 5 - 8 = \sqrt{33} \times \sqrt{33} \cos\theta$

$\cos\theta = -\dfrac{1}{3}, \quad \theta = 109.5^o$

Q3ci Let $\boldsymbol{u} = p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$ be the unit vector, $p > 0$.

$\therefore p^2 + q^2 + r^2 = 1 \qquad (1)$

$\boldsymbol{u} \bullet \overrightarrow{OA} = 0, \ \therefore 4p + q + 4r = 0 \qquad (2)$

$\boldsymbol{u} \bullet \overrightarrow{OC} = 0, \ \therefore -2p + 5q + -2r = 0 \qquad (3)$

Solve (2) and (3) simultaneously, $11q = 0$,

$\therefore q = 0$ and $r = -p$.

Substitute in (1), $p^2 + p^2 = 1$, $\therefore p = \dfrac{1}{\sqrt{2}}$ and

$r = -\dfrac{1}{\sqrt{2}}$.

Q3cii $\overrightarrow{OD} = 3\boldsymbol{i} + 4\boldsymbol{j} - \dfrac{1}{3}\boldsymbol{k}$

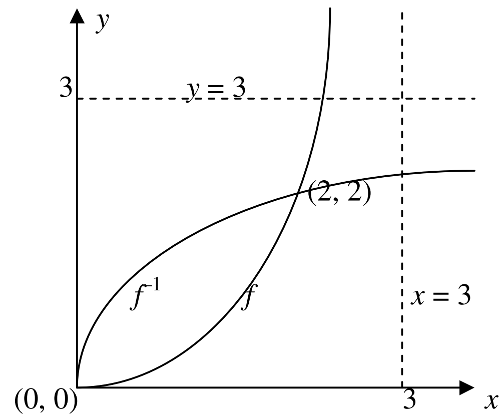
$\overline{DE}$ = scalar resolute of $\overrightarrow{OD}$ in the direction of $\boldsymbol{u}$

$= \overrightarrow{OD} \bullet \boldsymbol{u}$

$= 3 \times \dfrac{1}{\sqrt{2}} + \dfrac{1}{3} \times \dfrac{1}{\sqrt{2}} = \dfrac{5\sqrt{2}}{3}$.

Q4a $f(2) = -2 + 2\sec\left(\dfrac{\pi}{3}\right) = -2 + 2 \times 2 = 2$

Q4b

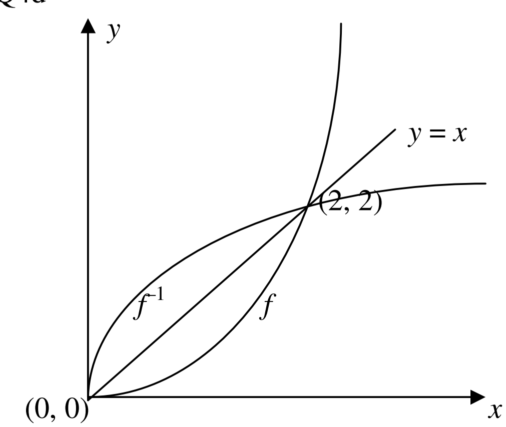


Q4c Let the equation of f be $y = -2 + 2\sec\left(\dfrac{\pi x}{6}\right)$, and $x \in [0,3)$, then the equation of f^{-1} is $x = -2 + 2\sec\left(\dfrac{\pi y}{6}\right)$.

$\therefore x + 2 = \dfrac{2}{\cos\left(\dfrac{\pi y}{6}\right)}$ or $\cos\left(\dfrac{\pi y}{6}\right) = \dfrac{2}{x+2}$.

$\therefore y = \dfrac{6}{\pi} Cos^{-1}\left(\dfrac{2}{x+2}\right), \ \therefore a = \dfrac{6}{\pi}$.

Q4d



$A = 2 \times \int_0^2 \left(\dfrac{6}{\pi} Cos^{-1}\left(\dfrac{2}{x+2}\right) - x \right) dx$ or

$= 2 \times \int_0^2 \left(x + 2 - 2\sec\left(\dfrac{\pi x}{6}\right) \right) dx$

$= 1.939$ (graphics calculator)

Q4e Let $u=\dfrac{1+\sin kx}{\cos kx}$ and $y=\log_e u$.

$$\frac{du}{dx}=\frac{(\cos kx)(k\cos kx)-(1+\sin kx)(-k\sin kx)}{\cos^2 kx}$$
$$=\frac{k\cos^2 kx+k\sin kx+k\sin^2 kx}{\cos^2 kx}$$
$$=\frac{k(1+\sin kx)}{\cos^2 kx}$$
$$\frac{dy}{du}=\frac{1}{u}=\frac{\cos kx}{1+\sin kx}$$
$$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{k}{\cos kx}=k\sec kx$$

Q4f

$$A=2\times\int_0^2\left(x+2-2\sec\left(\frac{\pi x}{6}\right)\right)dx$$
$$=2\times\int_0^2\left(x+2-\frac{12}{\pi}\times\frac{\pi}{6}\sec\left(\frac{\pi x}{6}\right)\right)dx$$
$$=2\times\int_0^2(x+2)dx-\frac{24}{\pi}\times\int_0^2\left(\frac{\pi}{6}\sec\left(\frac{\pi x}{6}\right)\right)dx$$
$$=2\times\left[\frac{(x+2)^2}{2}\right]_0^2-\frac{24}{\pi}\times\left[\log_e\left(\frac{1+\sin\left(\frac{\pi x}{6}\right)}{\cos\left(\frac{\pi x}{6}\right)}\right)\right]_0^2$$
$$=12-\frac{24}{\pi}\log_e\left(2+\sqrt{3}\right).$$

Q5a $\dfrac{dt}{dy}=\dfrac{1}{a}\times\dfrac{1}{100-y}$, $t=\dfrac{1}{a}\int\dfrac{dy}{100-y}$,

$at=-\log_e(100-y)+c$, $y=5$ when $t=0$,

$\therefore c=\log_e 95$ and $\therefore at=\log_e\left(\dfrac{95}{100-y}\right)$.

$\dfrac{100-y}{95}=e^{-at}$, $y=100-95e^{-at}$.

Q5b $y=10+Ae^{-b(t-T)}$, $\dfrac{dy}{dt}=-Abe^{-b(t-T)}$,

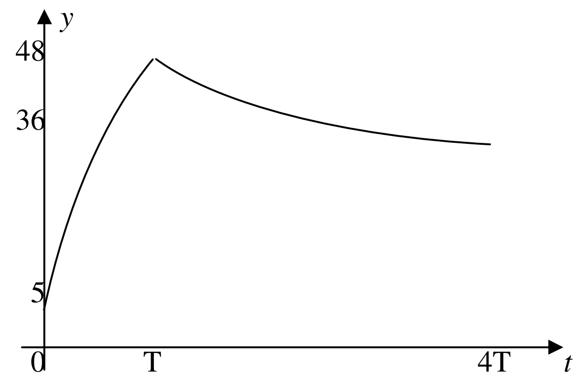
$$-b(y-10)=-b\left(Ae^{-b(t-T)}\right)$$
$$\therefore \frac{dy}{dt}=-b(y-10).$$

$y=48$ when $t=T$, $48=10+A$, $\therefore A=38$.

Q5c

$y=100-95e^{-at}$, $[0,T]$.

$y=10+38e^{-b(t-T)}$, $(T,4T]$



Q5d When $t=T$, $48=100-95e^{-aT}$,

$\therefore e^{-aT}=\dfrac{52}{95}$ and $aT=\log_e\left(\dfrac{95}{52}\right)$.

When $t=4T$, $36=10+38e^{-3bT}$

$\therefore e^{-3bT}=\dfrac{13}{19}$, and $3bT=\log_e\left(\dfrac{19}{13}\right)$.

$$\frac{aT}{3bT}=\frac{\log_e\left(\frac{95}{52}\right)}{\log_e\left(\frac{19}{13}\right)}$$
$$\frac{a}{b}=\frac{3\times\log_e\left(\frac{95}{52}\right)}{\log_e\left(\frac{19}{13}\right)}=4.76$$

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