Transformations in the x-y plane

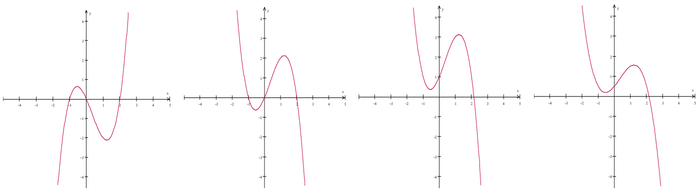
(Suitable for years 11 and 12)

A sequence of transformations of a relation or function can be inferred from an equation or a graph.

The lesson covers the following exercises:

(1) *Sketch the graph of a relation following a sequence of transformations*

Example 1 The first graph undergoes a sequence of transformations: Reflection in the x-axis; translate upwards by 1 unit; dilate from the x-axis by a factor of $\frac{1}{2}$. Sketch a sequence of graphs to show the transformations.



(2) *Find the equation of a relation after a sequence of transformations*

For reflection in the y-axis, place a negative sign in front of $\boldsymbol{x}$: $y = f(x) \to y = f(-x)$.

For reflection in the x-axis, place a negative sign in front of $\boldsymbol{y}$: $y = f(x) \to -y = f(x)$.

For translation of c units in the y-direction, add (down) or subtract (up) c from $\boldsymbol{y}$: $y = f(x) \to y \pm c = f(x)$.

For translation of b units in the x-direction, add (left) or subtract (right) b from $\boldsymbol{x}$: $y = f(x) \to y = f(x \pm b)$.

For dilation by a factor of a from the x-axis, change $\boldsymbol{y}$ to $\frac{y}{a}$:

$y = f(x) \to \frac{y}{a} = f(x)$.

For dilation by a factor of n from the y-axis, change $\boldsymbol{x}$ to $\frac{x}{n}$:

$y = f(x) \to y = f\left(\frac{x}{n}\right)$.

Example 2 The relation $(x-1)^2 + y^2 = 1$ undergoes a sequence of transformations: Reflection in the y-axis; translate to the left by 1 unit; dilate from the y-axis by a factor of $\frac{1}{2}$; dilate from the x-axis by a factor of 3. Find the equation of the transformed relation.

$(x-1)^2 + y^2 = 1$

$\to (-x-1)^2 + y^2 = 1$

$\to (-(x+1)-1)^2 + y^2 = 1$

$\to (-(2x+1)-1)^2 + y^2 = 1$

$\to (-(2x+1)-1)^2 + \left(\frac{y}{3}\right)^2 = 1$

The last equation can be simplified to $4(x+1)^2 + \frac{y^2}{9} = 1$.

(3) *Find the sequence of transformations from an equation*

The sequence of transformations depends on the form of the equation. If the equation is written with $(x \pm b)$ and $(y \pm c)$, translations are carried out last. Without the brackets translations are done before dilations and reflections.

Example 3 State the sequence of transformations changing $y = x^2$ to $-2y - 1 = \left(\frac{-x}{3} + 1\right)^2$.

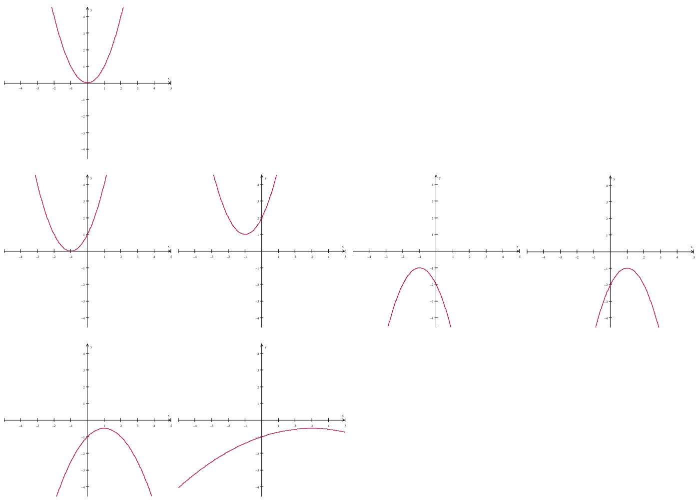
The sequence of transformations: Translation to the left by 1 unit; upward translation by 1 unit; reflection in the x-axis, reflection in the y-axis; dilation from the x-axis by a factor of $\frac{1}{2}$; dilation from the y-axis by a factor of 3.

If the equation is changed to the form $y = -\frac{1}{18}(x-3)^2 - \frac{1}{2}$ by transposition, the sequence of transformations is: Reflection in the x-axis; dilation from the x-axis by a factor of $\frac{1}{18}$; translation to the right by 3 units; downward translation by $\frac{1}{2}$ of a unit.

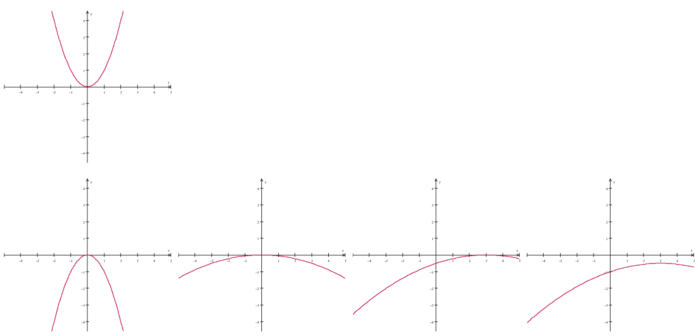
Note that one of the reflections and one of the dilations are gone, and the dilation from the x-axis and the translations are changed.

(4) *Sketch graph from an equation*

Example 4a Sketch a sequence of graphs changing $y = x^2$ to $-2y - 1 = \left(\frac{-x}{3} + 1\right)^2$.

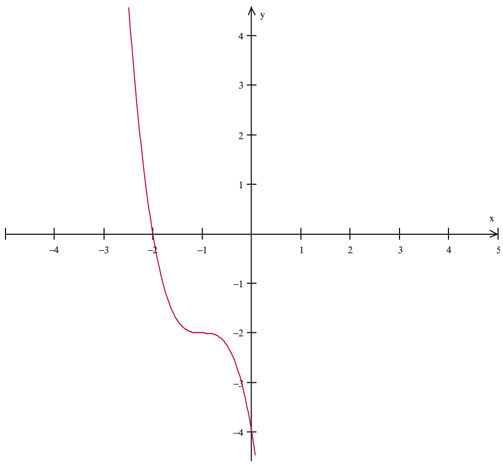


Example 4b Sketch a sequence of graphs changing $y = x^2$ to $-18\left(y + \frac{1}{2}\right) = (x-3)^2$.



(5) *Find equation from a graph*

Example 5 The following diagram is the graph of $y = x^3$ after a few transformations. Find the equation of the graph.



The stationary point of inflection is $(-1,-2)$.

The equation of the graph is $y = a(x+1)^3 - 2$.

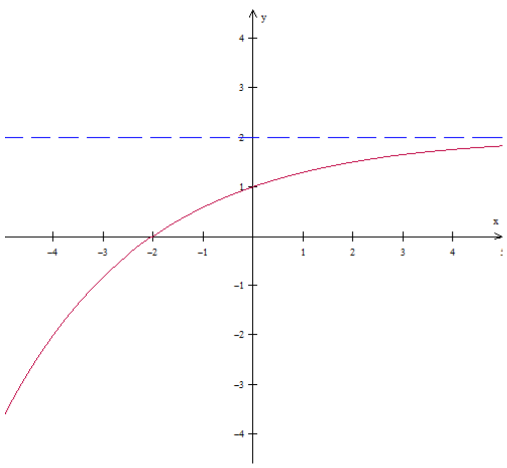
Use the y-intercept $(0,-4)$ to find a: $-4 = a(0+1)^3 - 2$, .: $a = -2$,

$y = -2(x+1)^3 - 2$

(6) *State the sequence of transformations from a graph*

In general, it would be easier to find the equation of the graph and then determine the sequence of transformations.

Example 6 List a sequence of transformations required to change the graph of $y = 2^x$ to the following graph.



The equation of the graph is in the form $y = a.2^{kx} + 2$.

y-intercept $(0,1)$: $1 = a.2^{k.0} + 2$, .: $a = -1$, .: $y = -1 \times 2^{kx} + 2$

x-intercept $(-2,0)$: $0 = -1 \times 2^{k(-2)} + 2$, .: $k = -\frac{1}{2}$

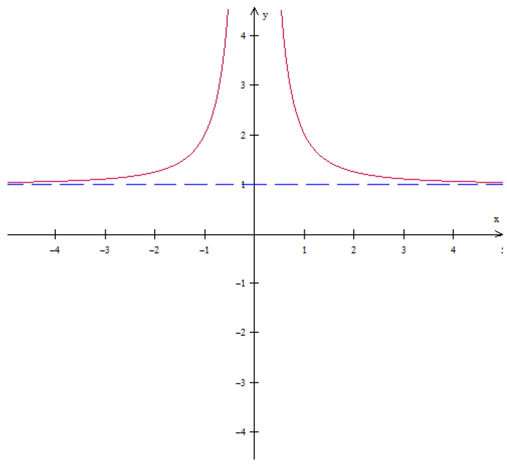
.: $y = -2^{-\frac{x}{2}} + 2$ or $-(y-2) = 2^{-\frac{x}{2}}$

Sequence of transformations: Reflection in both axes; dilation from the y-axis by a factor of 2; upward translation by 2 units.

Exercise:

1. Apply the following sequence of transformations to the given graph and sketch the resulting graph.

Translation to the left by 1 unit; reflection in the y-axis; dilation from the x-axis by a factor of 2; downward translation by 2 units



2. Consider $f(x) = 1 - \frac{\sqrt{2-3x}}{2}$ and $g(x) = \sqrt{x}$.

a. Sketch the graphs of $y = f(x)$ and $y = g(x)$.

b. Write a sequence of transformations that changes $f(x)$ to $g(x)$.

3. The following sequence of transformations is applied to the graph of $x^2 + y^2 = 4$.

Translate to the left by 1 unit; reflection in the y-axis; dilate from the y-axis by a factor of $\frac{1}{2}$; dilate from the x-axis by a factor of 2.

a. Sketch the resulting graph.

b. Find the equation of the resulting relation.

4. The following sequence of transformations is applied to the graph of $y = x(x-1)(x+2)+1$.

Translate to the left by 1 unit; reflection in the x-axis; dilate from the x-axis by a factor of 2; dilate from the y-axis by a factor of $\frac{1}{2}$.

a. Sketch the resulting graph.

b. Find the equation of the resulting relation.