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System of simultaneous linear equations

Here we consider only equations with unknowns x, y, z, … each to the power of 1 (linear). Usually the unknowns are placed on the left side of an equation and the constant on the right. Always arrange the unknowns in the same order in each equation. If an unknown is missing, insert it in the equation with coefficient 0.

The system of equations $y-x=3+z$, $2y-1=z$ *and* $5x+2=-3$ *can be rearranged to become*

$-x+y-z=3$ .... *(1)*

$0x+2y-z=1$ .... *(2)*

$5x+0y+0z=-5$ .... *(3)*

Solving simultaneous linear equations by matrix method

This is the preferred method if there are many unknowns in the equations.

Example 1 Solve simultaneous equations (1), (2) and (3) above by matrix method.

Matrix representation:

$$\begin{bmatrix}-1 & 1 & -1\\ 0 & 2 & -1\\ 5 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ 1\\ -5\end{bmatrix},\ \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-1 & 1 & -1\\ 0 & 2 & -1\\ 5 & 0 & 0\end{bmatrix}^{-1}\begin{bmatrix}3\\ 1\\ -5\end{bmatrix}=\begin{bmatrix}-1\\ -1\\ -3\end{bmatrix}$$

Geometric interpretation: The solution $(-1,-1,-3)$ *is the point of intersection of three planes represented by the three equations in 3-D space.*

Example 2 Solve the following system of simultaneous equations: $y-2x+z=2$, $-y-x+z=-1$, $-2y+4x-2z=1$

The terms are already in the same order.

$$\begin{bmatrix}1 & -2 & 1\\ -1 & -1 & 1\\ -2 & 4 & -2\end{bmatrix}\begin{bmatrix}y\\ x\\ z\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix},\ \begin{bmatrix}y\\ x\\ z\end{bmatrix}=\begin{bmatrix}1 & -2 & 1\\ -1 & -1 & 1\\ -2 & 4 & -2\end{bmatrix}^{-1}\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$$

$\begin{bmatrix}1 & -2 & 1\\ -1 & -1 & 1\\ -2 & 4 & -2\end{bmatrix}$ is a singular matrix and its inverse does not exist, .: matrix method cannot be used to solve the equations. In fact there are no solutions to these simultaneous equations.

Geometric interpretation: The three planes do not intersect at a point because two of the planes represented by $y-2x+z=2$ *and* $-2y+4x-2z=1$ *are parallel and non-overlapping.*

Example 3 Solve the following system of simultaneous equations: $y-2x+z=2$, $-y-x+z=-1$, $-2y+4x-2z=-4$

$\begin{bmatrix}1 & -2 & 1\\ -1 & -1 & 1\\ -2 & 4 & -2\end{bmatrix}\begin{bmatrix}y\\ x\\ z\end{bmatrix}=\begin{bmatrix}2\\ -1\\ -4\end{bmatrix}$. The same singular matrix, .: matrix method cannot be used to solve the equations.
However, solutions exist for these simultaneous equations.

Geometric interpretation: $y-2x+z=2$ *and* $-2y+4x-2z=-4$ *are two overlapping planes. The third plane* $-y-x+z=-1$ *intersects the overlapping planes at infinitely many points. Hence there are infinitely many solutions.*

Solving simultaneous linear equations by elimination and substitution

Example 4 Solve the system of simultaneous equations:

$y-2x+z=2$ .... (1)

$-y-x+z=-1$ .... (2)

$-2y+4x-2z=-4$ .... (3)

(1) and (3) are the same equation, .: we have 2 equations with 3 unknowns.

Eliminate y by (1) + (2): $-3x+2z=1$, .: $x=\frac{2z-1}{3}$

Let $z=c$ where $c\in R$, .: $x=\frac{2c-1}{3}$ .... (4)

Substitute (4) in (1): $y-2\left(\frac{2c-1}{3}\right)+c=2$, $y=\frac{c+4}{3}$

Solution set: $\left\{(x,y,z): x=\frac{2c-1}{3}, y=\frac{c+4}{3}, z=c, c\in R\right\}$

Note: If the square matrix in the matrix representation of a system of simultaneous linear equations is singular, its inverse does not exist. Either the system of simultaneous equations has no solutions or it has infinitely many solutions.

Example 5 The curve $y=ax^3+bx^2+cx+d$ passes through the points $(-2,26)$, $(-1,10)$, $(1,2)$ and $(2,-2)$. Find the values of a, b, c and d.

$(-2,26)\rightarrow -8a+4b-2c+d=26$

$(-1,10)\rightarrow -a+b-c+d=10$

$(1,2)\rightarrow a+b+c+d=2$

$(2,-2)\rightarrow 8a+4b+2c+d=-2$

$$\begin{bmatrix}-8 & 4 & -2 & 1\\ -1 & 1 & -1 & 1\\ 1 & 1 & 1 & 1\\ 8 & 4 & 2 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}26\\ 10\\ 2\\ -2\end{bmatrix},\ \begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}-8 & 4 & -2 & 1\\ -1 & 1 & -1 & 1\\ 1 & 1 & 1 & 1\\ 8 & 4 & 2 & 1\end{bmatrix}^{-1}\begin{bmatrix}26\\ 10\\ 2\\ -2\end{bmatrix}=\begin{bmatrix}-1\\ 2\\ -3\\ 4\end{bmatrix}$$

.: $a=-1$, $b=2$, $c=-3$, $d=4$

Example 6 Find **a** cubic polynomial function *f* that satisfies the conditions $f'(3)=0$, $f(3)=4$ and $f(10)=-1$. (This is an example from the Study Design)

Let $f(x)=ax^3+bx^2+cx+d$. $f'(x)=3ax^2+2bx+c$

There are only 3 pieces of information given to find 4 unknowns. Choose a number for one of the unknowns, say $a=1$.

$f(x)=x^3+bx^2+cx+d$ and $f'(x)=3x^2+2bx+c$

$f'(3)=0\rightarrow 6b+c=-27$

$f(3)=4\rightarrow 9b+3c+d=-23$

$f(10)=-1\rightarrow 100b+10c+d=-1001$

$$\begin{bmatrix}6 & 1 & 0\\ 9 & 3 & 1\\ 100 & 10 & 1\end{bmatrix}\begin{bmatrix}b\\ c\\ d\end{bmatrix}=\begin{bmatrix}-27\\ -23\\ -1001\end{bmatrix},\ \begin{bmatrix}b\\ c\\ d\end{bmatrix}=\begin{bmatrix}6 & 1 & 0\\ 9 & 3 & 1\\ 100 & 10 & 1\end{bmatrix}^{-1}\begin{bmatrix}-27\\ -23\\ -1001\end{bmatrix}\approx\begin{bmatrix}-16.102041\\ 69.612245\\ -86.918367\end{bmatrix}$$

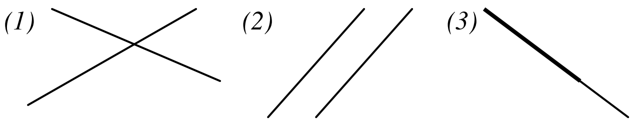
.: $f(x)\approx x^3-16.102041x^2+69.612245x-86.918367$

Note: If you have chosen a value for b, c or d instead of a, a different function will be obtained. The question is asking for **a** cubic polynomial function *f*, there are infinitely many functions satisfying the given conditions.

Unique solution, no solution, infinitely many solutions to simultaneous linear equations with two unknowns

Here we consider 2 linear equations with 2 unknowns. Each equation can be interpreted as a straight line in the Cartesian plane.

Three possibilities:



(1) Two non-parallel lines (different gradients) always intersect at a point, .: the two linear equations have a unique solution.

(2) Two separate parallel lines (same gradient but different y-intercepts) never intersect, .: the two linear equations have no solutions.

(3) Two overlapping lines (same gradient and y-intercept) have infinite number of intersections, .: the two linear equations have infinitely many solutions.

Example 7 Solve the simultaneous equations by (a) elimination method (b) matrix method. $y=3-2x$, $x-y=6$

(a) $2x+y=3$ ……… (1) $x-y=6$ ……… (2)
Eliminate y by (1) + (2): $3x=9$, $x=3$ ……… (3)
Substitute (3) in (2): $3-y=6$, $y=-3$

(b) $\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}3\\ 6\end{bmatrix}$, $\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}^{-1}\begin{bmatrix}3\\ 6\end{bmatrix}=\begin{bmatrix}3\\ -3\end{bmatrix}$

.: $x=3$ and $y=-3$

The two simultaneous equations have a unique solution.

Example 8 Determine whether the simultaneous equations have a unique solution, no solutions or infinitely many solutions.
(a) $3x-5y+1=0$, $2y+x=3$
(b) $x+3y-2=0$, $2y-3=-\frac{2x}{3}$ (c) $x=-0.5$, $y=-0.5$

(a) Write both equations in $y=mx+c$ form.

$3x-5y+1=0 \rightarrow y=\frac{3}{5}x+\frac{1}{5}$, $2y+x=3 \rightarrow y=-\frac{1}{2}x+\frac{3}{2}$

The gradients are different, .: they have a unique solution.

(b) $x+3y-2=0 \rightarrow y=-\frac{1}{3}x+\frac{2}{3}$,

$2y-3=-\frac{2x}{3} \rightarrow y=-\frac{1}{3}x+\frac{3}{2}$

Same gradient but different y-intercepts, .: no solutions.

(c) $x=-0.5$ is a vertical line and $y=-0.5$ is a horizontal line, .: they intersect at $(-0.5,-0.5)$.
The equations have a unique solution.

Example 9 (2006 VCAA Sample Exam 2 Version 2)
Find m such that the simultaneous equations $(m-2)x+3y=6$ and $2x+(m+2)y=m$ will have a unique solution.

Write both equations in $y=mx+c$ form.

$(m-2)x+3y=6 \rightarrow y=-\frac{m-2}{3}x+2$

$2x+(m+2)y=m \rightarrow y=-\frac{2}{m+2}x+\frac{m}{m+2}$

For a unique solution to exist, the gradients must be different.

.: $-\frac{m-2}{3}\neq-\frac{2}{m+2}$, .: $m^2-4\neq 6$, $m^2\neq 10$, $m\neq\pm\sqrt{10}$

.: $m\in R\setminus\{-\sqrt{10},\sqrt{10}\}$

Example 10 (2010 VCAA Exam 2)
Find m such that the simultaneous equations $(m-1)x+5y=7$ and $3x+(m-3)y=0.7m$ will have (a) infinitely many solutions (b) no solutions.

Write both equations in $y=mx+c$ form.

$(m-1)x+5y=7 \rightarrow y=-\frac{m-1}{5}x+\frac{7}{5}$

$3x+(m-3)y=0.7m \rightarrow y=-\frac{3}{m-3}x+\frac{0.7m}{m-3}$

(a) Infinitely many solutions: Same gradient AND y-intercept.
$\frac{m-1}{5}=\frac{3}{m-3}$ AND $\frac{7}{5}=\frac{0.7m}{m-3}$
.: $(m-1)(m-3)=15$ AND $10(m-3)=5m$
.: $m^2-4m-12=0$ AND $5m=30$
.: $m=-2$ or 6 AND $m=6$, .: $m=6$

(b) No solutions: Same gradient AND different y-intercepts.
$\frac{m-1}{5}=\frac{3}{m-3}$ AND $\frac{7}{5}\neq\frac{0.7m}{m-3}$
.: $m=-2$ or 6 AND $m\neq 6$, .: $m=-2$

Example 11 (2011 VCAA Exam 1) Consider the simultaneous linear equations $kx-3y=k+3$, $4x+(k+7)y=1$ where k is a real constant. (a) Find the value of k for which there are infinitely many solutions. (b) Find the values of k for which there is a unique solution.

$kx-3y=k+3 \rightarrow y=\frac{k}{3}x-\frac{k+3}{3}$

$4x+(k+7)y=1 \rightarrow y=-\frac{4}{k+7}x+\frac{1}{k+7}$

(a) Infinitely many solutions: Same gradient AND y-intercept.
$\frac{k}{3}=-\frac{4}{k+7}$ AND $-\frac{k+3}{3}=\frac{1}{k+7}$
.: $k(k+7)+12=0$ AND $(k+3)(k+7)+3=0$
$k^2+7k+12=0$ AND $k^2+10k+24=0$
.: $k=-4$ or -3 AND $k=-4$ or -6, .: $k=-4$

(b) A unique solution: Different gradients.
$\frac{k}{3}\neq-\frac{4}{k+7}$, .: $k\neq-4$ or -3, .: $k\in R\setminus\{-4,-3\}$

Example 12 Consider the simultaneous equations $px+3y=5$ and $5x+(p+2)y=p$. (a) Find the value(s) of p such that the equations have no solutions or infinitely many solutions. (b) Find the value(s) of p such that the equations have no solutions.

(a) In matrix form, $\begin{bmatrix}p & 3\\ 5 & p+2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}5\\ p\end{bmatrix}$

No solutions or infinitely many solutions, *determinant* = 0,
$p(p+2)-3\times5=0$, $p^2+2p-15=0$,
$(p+5)(p-3)=0$, .: $p=-5$ or 3

(b) $y=-\frac{p}{3}x+\frac{5}{3}$, $y=-\frac{5}{p+2}x+\frac{p}{p+2}$

No solutions: Same gradient and different y-intercepts.
$-\frac{p}{3}=-\frac{5}{p+2}$ AND $\frac{5}{3}\neq\frac{p}{p+2}$
.: $p=-5$ or 3 AND $p\neq-5$
.: $p=3$

Exercise: Next page

Q1 Solve the system of simultaneous equations by matrix method. $x+z-y=-1$, $-x+2y+2z=4$, $2x+3z=-1$

Q2 Solve the system of simultaneous equations by substitution/elimination. $x-2y-z=4$, $2x-y+z=2$, $x+y-2z=2$

Q3 Solve the system of simultaneous equations by substitution/elimination. $x+y+z=1$, $x-y-z=1$, $-2x+y+z=-2$

Q4 Solve the system of simultaneous equations if $x=z$.
$-x+y-3z=2$, $\frac{2}{3}x-\frac{2}{3}y+z=-\frac{4}{3}$,
$0.3x-0.3y+0.9z=-0.6$

Q5 The curve $y=ax^2+bx+c$ passes through the points with coordinates $(-1,0)$, $(1,-2)$ and $(2,3)$. Find the values of *a*, *b* and *c*.

Q6 The rule $f(x)=ax^2+bx+c$ satisfies the following conditions. $f'(1)=0$, $f(1)=\frac{5}{2}$ and $f(-1)=\frac{1}{2}$. Find the values of *a*, *b* and *c*.

Q7 Find $m\in R$ such that the simultaneous equations $(m-2)x+3y=6$ and $2x+(m-3)y=m-1$ will have no solution.

Q8 Find $m\in R$ such that the simultaneous equations $mx+12y=24$ and $3x+my=m$ will have a unique solution.

Q9 Find $a\in R$ such that the simultaneous equations $ax+3y=0$ and $2x+(a+1)y=0$ will have infinitely many solutions.

Q10 Find $k\in R$ such that the simultaneous equations $kx-3y=0$ and $5x-(k+2)y=0$ will have a unique solution.

Numerical, algebraic and worded answers: 1. $x=-2$, $y=0$, $z=1$ 2. $x=1$, $y=-1$, $z=-1$ 3. $x=1$, $y=c$, $z=-c$ where $c\in R$ 9. 9. $a=-3$ or 2
4. $x=c$, $y=4c+2$, $z=c$ where $c\in R$ 5. $a=2$, $b=-1$, $c=-3$ 6. $a=-0.5$, $b=1$, $c=2$ 7. $m=0$ 8. $m\in R\setminus\{-6,6\}$ 10. $k\in R\setminus\{-5,3\}$