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Factorisation (cont.)

Grouping *A four term cubic polynomial is grouped into two terms and two terms. The two groups are then factorised separately. Correct grouping will give a common factor in the two groups.*

Example 1 Factorise $2x^3-3x^2-32x+48$ over R.

Group the first two terms and the last two.

$2x^3-3x^2-32x+48=\left(2x^3-3x^2\right)-\left(32x-48\right)$

$=x^2(2x-3)-16(2x-3)=(2x-3)\left(x^2-16\right)$

$=(2x-3)\left(x^2-4^2\right)=(2x-3)(x-4)(x+4)$.

Note: In this case it is also possible to proceed by grouping the first and the third terms; and the second and the last terms.

$2x^3-3x^2-32x+48=\left(2x^3-32x\right)-\left(3x^2-48\right)$

$=2x\left(x^2-16\right)-3\left(x^2-16\right)=\left(x^2-16\right)(2x-3)$

$=(x-4)(x+4)(2x-3)$

Example 2 Factorise $8x^3-2x^2+x-1$ over R.

$8x^3-2x^2+x-1=\left(8x^3-1\right)-\left(2x^2-x\right)=\left((2x)^3-1^3\right)-\left(2x^2-x\right)$

$=(2x-1)\left((2x)^2+(2x)1+1^2\right)-x(2x-1)$

$=(2x-1)\left(4x^2+2x+1\right)-x(2x-1)$

$=(2x-1)\left(4x^2+2x+1-x\right)=(2x-1)\left(4x^2+x+1\right)$.

The factor theorem *Consider a polynomial $P(x)$ that has $x-\alpha$ as a linear factor. Then $P(x)=(x-\alpha)Q(x)$, where $Q(x)$ is a polynomial one degree lower than $P(x)$, obtained by expanding and comparing coefficients, or dividing $P(x)$ by $x-\alpha$. Replacing x by α in $P(x)=(x-\alpha)Q(x)$, $P(\alpha)=(\alpha-\alpha)Q(\alpha)$. Hence $P(\alpha)=0$.*

Conversely, for any polynomial $P(x)$, if $P(\alpha)=0$, then $x-\alpha$ is a factor of $P(x)$. This statement is known as the factor theorem, and can be used to find the linear factors of a polynomial if other methods failed.

The factor theorem is best used for polynomials with linear factors of rational coefficients. The value(s) of α is found by trial and error. The possible values of α for trying depend on the first and last coefficients of

$P(x)=a_0x^n+a_1x^{n-1}+......+a_{n-1}x+a_n$, $\alpha=\pm\dfrac{a.factor.of.a_n}{a.factor.of.a_0}$.

If all these α values give $P(\alpha)\neq 0$, it does not necessarily mean that $P(x)$ has no linear factors, because the coefficients of the linear factor(s) may be irrational.

Example 3 Use the factor theorem to find a linear factor of $3x^3-2x^2-7x-2$, then find the quadratic factor and hence all the linear factors.

Let $P(x)=3x^3-2x^2-7x-2$. The possible values of α for testing are $\alpha=\pm\dfrac{1,2}{1,3}$, i.e. $\alpha=\pm\dfrac{1}{3},\pm\dfrac{2}{3},\pm1,\pm2$.

$P(1)=3(1)^3-2(1)^2-7(1)-2\neq 0$.

$P(-1)=3(-1)^3-2(-1)^2-7(-1)-2=0$, $\therefore x+1$ is a factor.

Divide $P(x)$ by $x+1$ to find the quadratic factor.

$$
\begin{array}{r}
3x^2-5x-2 \\
x+1\,\big)\ 3x^3-2x^2-7x-2 \\
-\left(3x^3+3x^2\right) \qquad\qquad \\
\hline
-5x^2-7x \qquad \\
-\left(-5x^2-5x\right) \qquad \\
\hline
-2x-2 \\
-(-2x-2) \\
\hline
0
\end{array}
$$

Hence $P(x)=(x+1)\left(3x^2-5x-2\right)=(x+1)(3x+1)(x-2)$.

Another possibility:

$P\left(-\dfrac{1}{3}\right)=3\left(-\dfrac{1}{3}\right)^3-2\left(-\dfrac{1}{3}\right)^2-7\left(-\dfrac{1}{3}\right)-2=0$, $\therefore x+\dfrac{1}{3}$ is a factor.

$$
\begin{array}{r}
3x^2-3x-6 \\
x+\tfrac{1}{3}\,\big)\ 3x^3-2x^2-7x-2 \\
-\left(3x^3+x^2\right) \qquad\qquad \\
\hline
-3x^2-7x \qquad \\
-\left(-3x^2-x\right) \qquad \\
\hline
-6x-2 \\
-(-6x-2) \\
\hline
0
\end{array}
$$

Hence $P(x)=\left(x+\dfrac{1}{3}\right)\left(3x^2-3x-6\right)=3\left(x+\dfrac{1}{3}\right)\left(x^2-x-2\right)$

$=3\left(x+\dfrac{1}{3}\right)(x-2)(x+1)$. It is equivalent to the previous result.

Example 4 Given $x+1$ is a factor of $3x^4+x^3-9x^2-9x-2$, find the cubic factor $Q(x)$ such that

$$3x^4+x^3-9x^2-9x-2=(x+1)Q(x).$$

Let $Q(x)=ax^3+bx^2+cx+d$, then

$3x^4+x^3-9x^2-9x-2=(x+1)\left(ax^3+bx^2+cx+d\right)$.

Expand to obtain $3x^4+x^3-9x^2-9x-2$

$=ax^4+(a+b)x^3+(b+c)x^2+(c+d)x+d$.

Compare the coefficients on both sides, $a=3$, $a+b=1$, $b+c=-9$, $c+d=-9$ and $d=-2$. Hence $b=-2$, $c=-7$.

$\therefore Q(x)=3x^3-2x^2-7x-2$.

This example illustrates an alternative method in finding $Q(x)$ to long division shown in the last example.

Example 5 Use the factor theorem to find the linear factors of $x^4-x^3+x^2-3x+2$.

Let $P(x)=x^4-x^3+x^2-3x+2$. Test $\alpha=\pm1,\pm2$.

$P(-1)=(-1)^4-(-1)^3+(-1)^2-3(-1)+2\neq 0$

$P(1)=(1)^4-(1)^3+(1)^2-3(1)+2=0$, $\therefore x-1$ is a factor.

Hence $P(x)=(x-1)Q(x)$.

Use long division or by comparing coefficients to find $Q(x)=x^3+x-2$.

Hence $P(x)=(x-1)(x^3+x-2)$. Use the factor theorem on $Q(x)=x^3+x-2$ to find its linear factor.
Test $\alpha=\pm1,\pm2$.
$Q(1)=(1)^3+(1)-2=0$, $\therefore x-1$ is a factor of $Q(x)$.
Hence $P(x)=(x-1)(x-1)T(x)$.
$T(x)$ is quadratic and found by long division of $P(x)$ by the expansion of $(x-1)(x-1)$, or comparison of coefficients as discussed in example 4, $T(x)=x^2+x+2$.
$\therefore P(x)=(x-1)(x-1)(x^2+x+2)$.

Example 6 (2011 VCAA Exam 2) If $x+a$ is a factor of $4x^3-13x^2-ax$, where $a\in R\setminus\{0\}$, find the value of a.

Let $f(x)=4x^3-13x^2-ax$.
Since $x+a$ is a factor, $f(-a)=4(-a)^3-13(-a)^2-a(-a)=0$
$\therefore -4a^3-13a^2+a^2=0$, $-4a^3-12a^2=0$, $-4a^2(a+3)=0$.
Since $a\neq0$, $\therefore a+3=0$, $a=-3$

Alternative method: Since $a\in R\setminus\{0\}$, $a\neq0$, $\therefore x+a$ is a different factor from x.
$\therefore 4x^3-13x^2-ax=x(4x^2-13x-a)$
$=x(x+a)(4x-1)=x(4x^2+(4a-1)x-a)$
$\therefore 4a-1=-13$, $a=-3$

The remainder theorem *When a polynomial $P(x)$ is divided by a linear binomial $\alpha x-\beta$, the remainder can be found quickly without actually carrying out the division.*

Since $Q(x)$ *Quotient*

$\alpha x-\beta)\overline{\;P(x)\;}$ *Dividend (Polynomial)*

Divisor R *Remainder*

$\therefore P(x)=(\alpha x-\beta)Q(x)+R$.

When $x=\frac{\beta}{\alpha}$, $P\left(\frac{\beta}{\alpha}\right)=\left(\alpha\left(\frac{\beta}{\alpha}\right)-\beta\right)Q(x)+R=R$,

i.e. the remainder $R=P\left(\frac{\beta}{\alpha}\right)$ *when* $P(x)$ *is divided by* $\alpha x-\beta$.
This is known as the remainder theorem.

Example 7 Find the remainder when $P(x)=2x^4-x^3+5x-11$ is divided by (i) $x+5$, (ii) $2x-3$, (iii) $x-2a$.

(i) $R=P(-5)=2(-5)^4-(-5)^3+5(-5)-11=1339$

(ii) $R=P\left(\frac{3}{2}\right)=2\left(\frac{3}{2}\right)^4-\left(\frac{3}{2}\right)^3+5\left(\frac{3}{2}\right)-11=\frac{13}{4}$

(iii)
$R=P(2a)=2(2a)^4-(2a)^3+5(2a)-11=32a^4-8a^3+10a-11$

Example 8 Given that $x-2$ is a factor of $3x^3+px^2+qx-2$ and the remainder is -20 when the cubic polynomial is divided by $x+2$. Find the values of p and q.

Use the factor theorem and the remainder theorem to set up two simultaneous equations for p and q.
Let $P(x)=3x^3+px^2+qx-2$.
$x-2$ is a factor of $P(x)$, $\therefore P(2)=0$,
$\therefore 3(2)^3+p(2)^2+q(2)-2=0$, $\therefore 2p+q=-11$ ……(1)

The remainder is -20 when divided by $x+2$, $\therefore P(-2)=-20$,
$\therefore 3(-2)^3+p(-2)^2+q(-2)-2=-20$, $\therefore 2p-q=3$ ……(2)

Solve equations (1) and (2) for p and q.
(1) + (2), $4p=-8$, $\therefore p=-2$
(1) – (2), $2q=-14$, $\therefore q=-7$

Solving polynomial equations

In general use the factorisation methods to change a polynomial $P(x)$ to factorised form to find the exact solutions to $P(x)=0$, or use CAS to find the approximate solutions.

Example 9 (2006 VCAA Exam 2) Find the exact coordinates of the intercepts of $f(x)=0.25x^3-2.5x^2+4.25x+7$ with the axes.

$f(0)=7$, $\therefore$ y-intercept is $(0,7)$
x-intercepts: $f(x)=0.25x^3-2.5x^2+4.25x+7=0$
$f(x)=0.25(x^3-10x^2+17x+28)=0$
Factor theorem: $f(-1)=0.25(-1-10-17+28)=0$
$\therefore x+1$ is a factor
$\therefore f(x)=0.25(x+1)(x^2+px+28)=0.25(x^3+(p+1)x^2+...)$
Compare the coefficient of x^2, $p+1=-10$, $p=-11$
$\therefore f(x)=0.25(x+1)(x^2-11x+28)=0.25(x+1)(x-4)(x-7)=0$
$\therefore x=-1, 4, 7$
The x-intercepts are $(-1,0)$, $(4,0)$ and $(7,0)$.

Example 10 (2009 VCAA Exam 2)
The curve $y=\frac{1}{200}(x^3-6x^2+16)$ cuts the x-axis at $(2,0)$. Find the other x-intercepts.

Since $(2,0)$ is one of the x-intercepts, $\therefore x-2$ is a factor.
$y=\frac{1}{200}(x^3-6x^2+16)=\frac{1}{200}(x-2)(x^2+px-8)$
$=\frac{1}{200}(x^3+(p-2)x^2-(2p+8)x+16)$
Compare the coefficient of x^2, $p-2=-6$, $p=-4$.
Let $y=\frac{1}{200}(x-2)(x^2-4x-8)=0$.
$\therefore x^2-4x-8=0$, $x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-8)}}{2(1)}=2\pm2\sqrt{3}$
The other x-intercepts are $(2-2\sqrt{3},0)$ and $(2+2\sqrt{3},0)$.

Example 11 The curves $y=x^4-ax^2$ and $y=x^3+9x$ intersect at $x=a$ where $a\neq0$. Find the coordinates of all the intersections.

At the intersections $x^4-ax^2=x^3+9x$
$x^4-x^3-ax^2-9x=0$, $x(x^3-x^2-ax-9)=0$
$\therefore x=0$ or $x^3-x^2-ax-9=0$
$\therefore a^3-a^2-a^2-9=0$, $a^3-2a^2-9=0$, $\therefore a=3$ by the Factor Theorem.
$\therefore x^3-x^2-3x-9=0$, $(x-3)(x^2+2x+3)=0$ by comparing coefficients.
x^2+2x+3 cannot be factorised further because $\Delta<0$.
$\therefore$ There are only 2 intersections, their coordinates are:
$x=0$, $y=0$, $(0,0)$
$x=a=3$, $y=54$, $(3,54)$

Exercise: Next page

Q1 Factorise $3x^3 - x^2 - 2x + 24$ over R.

Q2 Factorise $x^3 - 2x^2 - 16x + 32$ over R.

Q3 Use one application of the factor theorem to find the linear factors of $x^3 - 13x^2 + 51x - 63$.

Q4 Find $Q(x)$ such that $x^4 + 4x^3 - 4x - 1 = (x-1)Q(x)$.

Q5 Use two applications of the factor theorem to find the linear factors of $x^4 - x^3 - 6x^2 + 4x + 8$.

Q6 If $x + a$ is a factor of $4x^3 - 9x^2 - ax$, where $a \in R \setminus \{0\}$, find the value of a.

Q7 Find the remainder when $P(x) = 2x^4 - 3x^3 + 7x - 11$ is divided by $2x - 1$.

Q8 Given that $x - 3$ is a factor of $2x^3 + ax^2 + bx - 6$ and the remainder is -12 when the cubic polynomial is divided by $x + 1$. Find the values of a and b.

Q9 Use the factor theorem to solve $2x^3 + 5x^2 + x - 2 = 0$.

Q10 The curves $y = x^4 - x^2$ and $y = x^3 - ax$ intersect at $x = a$ where $a \neq 0$. Find the coordinates of all the intersections.

Numerical, algebraic and worded answers: 1. $(x+2)(3x^2 - 7x + 12)$ 2. $(x-2)(x-4)(x+4)$ 3. $(x-3)^2(x-7)$ 4. $Q(x) = x^3 + 5x^2 + 5x + 1$ 5. $(x-2)^2(x+2)(x+1)$ 6. $a = -2$ 7. $-\frac{83}{8}$ 8. $a = -5$, $b = -1$ 9. $x = -1$, $-\frac{1}{2}$, 2 10. $(-1,0)$, $(0,0)$, $(1,0)$