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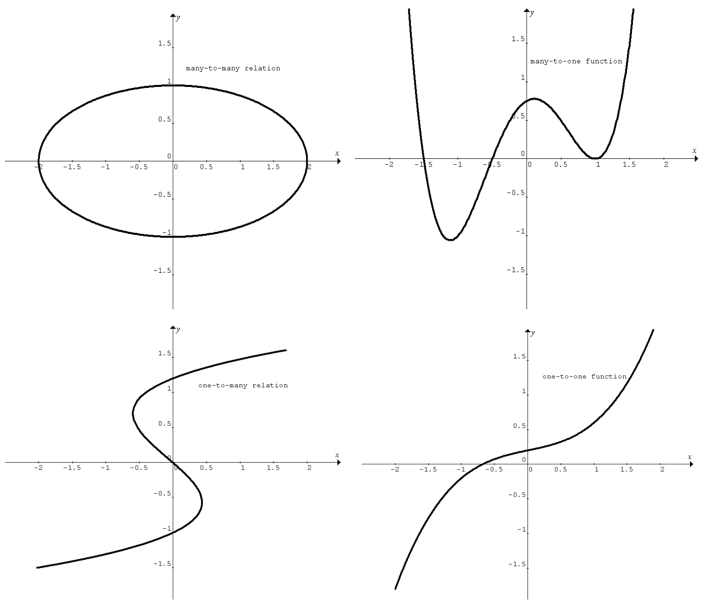
Free maths self-help revision

Relations and functions

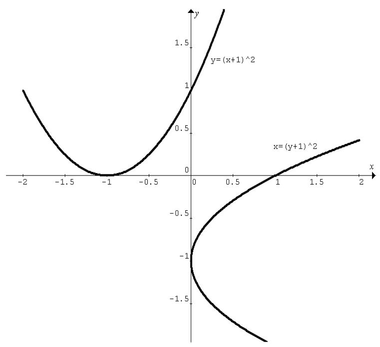
A relation is a set of ordered pairs (points). A function is a relation such that no two points have the same x-coordinate. Use the ***vertical line test*** *to determine whether a relation is a function (cuts through only one point) or not (cuts through more than one point). If it is a function, then it is either a* ***many-to-one function*** *or a* ***one-to-one function****. If it is not a function, then it is either a* ***many-to-many relation*** *or a* ***one-to-many relation****.*



Inverse functions

Every relation has an inverse that may or may not be a function. If a relation is a one-to-many relation, then its inverse is a many-to-one function. If a relation is a one-to-one function, then its inverse is also a function (a one-to-one function). If a relation is a many-to-many relation or many-to-one function, then its inverse is not a function.

Example 1 The following two graphs show the original relation $y=(x+1)^2$ which is a many-to-one function, and its inverse $x=(y+1)^2$ which is not a function.



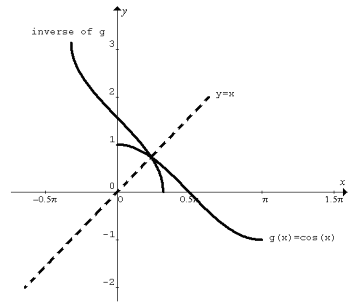
Example 2 The relations $y=\sin x$ for $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $y=x^2$ for $x\ge 0$ and $y=x^2$ for $x<0$ are one-to-one functions, .: the inverses are also one-to-one functions.

If a relation is a function, ***function notations*** *can be used to represent it, e.g.* $y=x^2$ *for* $x<0$, $f:R^-\to R, f(x)=x^2$*. Since its inverse* $y=-\sqrt{x}$ *is also a function, use* f^{-1} *to denote inverse function,* $f^{-1}:R^+\to R, f^{-1}(x)=-\sqrt{x}$.

Example 3 Restrict the domain of $y=\cos x$ to produce a one-to-one function. Use function notations to represent it and its inverse function.

There are infinitely many possible restrictions of the domain, e.g. $[0,\pi]$. Let the function with this restriction be $g:[0,\pi]\to R, g(x)=\cos x$.

The following graphs show that *g* and g^{-1} are both one-to-one functions. The graph of g^{-1} is obtained by reflecting *g* in the line $y=x$.



g has a range of $[-1,1]$, this becomes the domain of g^{-1}.
The inverse of $y=\cos x$ is $x=\cos y$, the equivalent form of $x=\cos y$ is $y=\cos^{-1}x$, .: $g^{-1}:[-1,1]\to R, g^{-1}(x)=\cos^{-1}x$.

Example 4 Given a function with equation $y=\frac{1}{2}\log_e(3x-1)+\frac{5}{2}$, find the domain, range and equation of its inverse with *y* as the subject.

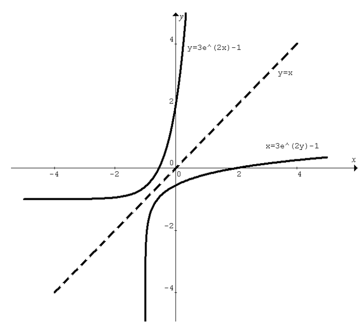
The given function is defined for $3x-1>0$, i.e. $x>\frac{1}{3}$, .: its domain is $\left(\frac{1}{3},\infty\right)$. Its range is *R*. Hence the domain and range of its inverse are *R* and $\left(\frac{1}{3},\infty\right)$ respectively.

The inverse of $y=\frac{1}{2}\log_e(3x-1)+\frac{5}{2}$ is $x=\frac{1}{2}\log_e(3y-1)+\frac{5}{2}$.

Transpose to make *y* the subject: $x-\frac{5}{2}=\frac{1}{2}\log_e(3y-1)$, $2x-5=\log_e(3y-1)$, the equivalent form is $3y-1=e^{2x-5}$, $3y=e^{2x-5}+1$, $y=\frac{1}{3}\left(e^{2x-5}+1\right)$ or $\frac{1}{3}e^{2x-5}+\frac{1}{3}$.

Example 5 (2006 VCAA Exam 1)
Given $f:R\to R, f(x)=3e^{2x}-1$. Find f^{-1}.

f is a one-to-one function, .: f^{-1} exists. The equation for graphing *f* is $y=3e^{2x}-1$. f has $(-1,\infty)$ as its range, .: the domain of f^{-1} is $(-1,\infty)$. The graph of f^{-1} is the reflection of *f* in the line $y=x$.



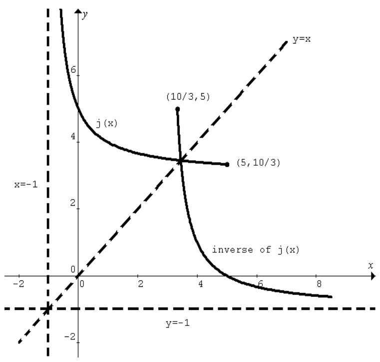
The equation of f^{-1} is $x=3e^{2y}-1$. Transpose to make *y* the subject: $x+1=3e^{2y}$, $\frac{x+1}{3}=e^{2y}$, the equivalent form is $2y=\log_e\left(\frac{x+1}{3}\right)$, $y=\frac{1}{2}\log_e\left(\frac{x+1}{3}\right)$.

Hence $f^{-1}:(-1,\infty)\to R, f^{-1}(x)=\frac{1}{2}\log_e\left(\frac{x+1}{3}\right)$.

Example 6 Given $j:(-1,5]\to R, j(x)=\frac{2}{x+1}+3$, find the domain, range, asymptote(s) and equation of its inverse.

Consider the given function *j*: Domain $(-1,5]$; range $\left[\frac{10}{3},\infty\right)$; equation $y=\frac{2}{x+1}+3$; vertical asymptote $x=-1$.
(The horizontal asymptote does not exist because the function has an end point at $x=5$, $y=\frac{2}{5+1}+3=\frac{10}{3}$)

The inverse of *j*: Domain $\left[\frac{10}{3},\infty\right)$; range $(-1,5]$; horizontal asymptote $y=-1$; equation $x=\frac{2}{y+1}+3$. Make *y* the subject by transposition: $x-3=\frac{2}{y+1}$, $y+1=\frac{2}{x-3}$, $y=\frac{2}{x-3}-1$.



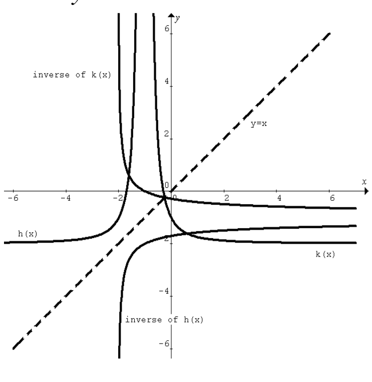
Example 7 Split $y=\frac{1}{(x+1)^2}-2$ into two one-to-one functions *h* and *k*, and find h^{-1} and k^{-1}.

$y=\frac{1}{(x+1)^2}-2$ is a many-to-one function and it is symmetrical about the vertical asymptote $x=-1$. Restrict the function to domain $(-\infty,-1)$ to form one-to-one function *h*. Restrict the function to domain $(-1,\infty)$ to form one-to-one function *k*.

Hence $h:(-\infty,-1)\to R, h(x)=\frac{1}{(x+1)^2}-2$

and $k:(-1,\infty)\to R, k(x)=\frac{1}{(x+1)^2}-2$.

The graphs of h^{-1} and k^{-1} are obtained by reflecting the graphs of *h* and *k* in the line $y=x$.



Both *h* and *k* have the same horizontal asymptote $y=-2$ and .: the same range $(-2,\infty)$.

.: h^{-1} and k^{-1} have the same vertical asymptote $x=-2$ and domain $(-2,\infty)$.

The inverse of $y=\frac{1}{(x+1)^2}-2$ is $x=\frac{1}{(y+1)^2}-2$.

The equivalent relation to $x=\frac{1}{(y+1)^2}-2$ is obtained by transposition making *y* the subject.

$x=\frac{1}{(y+1)^2}-2$, $x+2=\frac{1}{(y+1)^2}$, $(y+1)^2=\frac{1}{x+2}$, $y+1=\pm\frac{1}{\sqrt{x+2}}$, $y=\pm\frac{1}{\sqrt{x+2}}-1$.

Hence $h^{-1}:(-2,\infty)\to R, h^{-1}(x)=-\frac{1}{\sqrt{x+2}}-1$

and $k^{-1}:(-2,\infty)\to R, k^{-1}(x)=\frac{1}{\sqrt{x+2}}-1$.

Example 8 Find the domain and equation of the inverse of $y=\frac{1}{9}\left(x-\frac{1}{3}\right)^2+1$, where $x\ge-\frac{8}{3}$.

The given function has an end point $\left(-\frac{8}{3},2\right)$, and a turning point $\left(\frac{1}{3},1\right)$ that is the lowest point of the function. Hence the range is $[1,\infty)$, not $[2,\infty)$. .: the domain of the inverse is $[1,\infty)$.

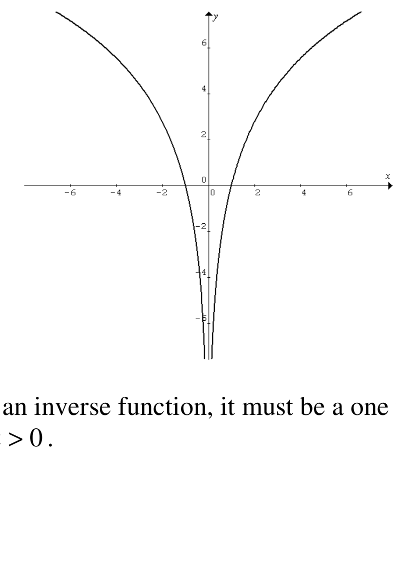
Equation of the inverse is $x=\frac{1}{9}\left(y-\frac{1}{3}\right)^2+1$.

Make *y* the subject: $x-1=\frac{1}{9}\left(y-\frac{1}{3}\right)^2$, $9(x-1)=\left(y-\frac{1}{3}\right)^2$, $y-\frac{1}{3}=\pm\sqrt{9(x-1)}$, $y-\frac{1}{3}=\pm3\sqrt{x-1}$, $y=\pm3\sqrt{x-1}+\frac{1}{3}$.

Note: (1) The inverse is not a function. (2) The inverse has an end point $\left(2,-\frac{8}{3}\right)$.

Example 9 (2006 VCAA Exam 2) Find *a* such that the function $f:[a,\infty)\to R,\ f(x)=\log_e\left(x^4\right)$ will have an inverse function.

The graph of $y=\log_e\left(x^4\right)$ is a many to one function.



For *f* to have an inverse function, it must be a one to one function, .: $a>0$.

Exercise: Next page

Q1 Restrict the domain of $y = \tan x$ to produce a one-to-one function. Use function notations to represent the function and its inverse function.

Q2 Restrict the domain of $y = \frac{2}{x^2}$ to produce a one-to-one function. Use function notations to represent the function and its inverse function.

Q3 Given a function with equation $y = \frac{1}{2}\sqrt{x-1} + \frac{5}{2}$, find the domain, range and equation of its inverse with *y* as the subject.

Q4 Given $f : R \rightarrow R, f(x) = 2e^{-2x} - 3$. Find the rule of f^{-1}, its domain and asymptote(s).

Q5 Given $f : (-2,4] \rightarrow R, f(x) = \frac{1}{x+2} - 1$, find the domain, range, asymptote(s) and equation of its inverse.

Q6 Find *a* such that the function $f : (-\infty, a) \rightarrow R,\ f(x) = \log_e|3-x|$ will have an inverse function.

Q7 Split $y = |a-x| - 1$, where *a* is a constant, into two one-to-one functions *h* and *k*, and find h^{-1} and k^{-1}.

Numerical, algebraic and worded answers:

1. E.g. $f : \left[0, \frac{\pi}{2}\right) \rightarrow R, f(x) = \tan x$, $f^{-1} : [0,\infty) \rightarrow R, f^{-1}(x) = \tan^{-1} x$

2. E.g. $f : (0,\infty) \rightarrow R, f(x) = \frac{2}{x^2}$, $f^{-1} : (0,\infty) \rightarrow R, f^{-1}(x) = \sqrt{\frac{2}{x}}$

3. Range: $[1,\infty)$, domain: $\left[\frac{5}{2},\infty\right)$, $y = 4\left(x - \frac{5}{2}\right)^2 + 1$

4. $f^{-1}(x) = \log_e \sqrt{\frac{2}{x+3}}$, domain: $(-3,\infty)$, asymptote: $x = -3$

5. Domain: $\left[-\frac{5}{6},\infty\right)$, range: $(-2,4]$, asymptote: $y = -2$, $y = \frac{1}{x+1} - 2$

6. $a < 3$

7. $h : (-\infty, a) \rightarrow R, h(x) = a - x - 1$, $k : [a,\infty) \rightarrow R, h(x) = x - a - 1$,
$h^{-1} : [-1,\infty) \rightarrow R, h(x) = a - x - 1$,
$k^{-1} : [-1,\infty) \rightarrow R, h(x) = x + a + 1$