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Free maths self-help revision

Inverse functions 'undo' each other

When a one-to-one function f and its inverse function f^{-1} *are used to form composite function* $f^{-1}\circ f$ *or* $f\circ f^{-1}$*, then* $f^{-1}\circ f(x)=x$, $f\circ f^{-1}(x)=x$*, i.e. they undo each other.*

Example 1 Given $f(x)=(x-2)^3$, find $f^{-1}(x)$. Show that they undo each other.

Equation of *f* is $y=(x-2)^3$, .: equation of f^{-1} is $x=(y-2)^3$, $y=\sqrt[3]{x}+2$, $f^{-1}(x)=\sqrt[3]{x}+2$.

$$f^{-1}\circ f(x)=f^{-1}(f(x))=\sqrt[3]{f(x)}+2=\sqrt[3]{(x-2)^3}+2=x$$
$$f\circ f^{-1}(x)=f(f^{-1}(x))=(f^{-1}(x)-2)^3=(\sqrt[3]{x})^3=x.$$

Example 2 Given $g(x)=\log_e x$, state $g^{-1}(x)$, find $g^{-1}\circ g(x)$ and $g\circ g^{-1}(x)$, and simplify.

$$g^{-1}(x)=e^x,\ g^{-1}\circ g(x)=g^{-1}(g(x))=e^{g(x)}=e^{\log_e x}=x.$$
$$g\circ g^{-1}(x)=g(g^{-1}(x))=\log_e e^x=x.$$

Example 3 Simplify $\sqrt{e^{2\log_e(x+1)-\log_e(x-1)^2}}$.

$$\sqrt{e^{2\log_e(x+1)-\log_e(x-1)^2}}=\left(e^{\log_e(x+1)^2-\log_e(x-1)^2}\right)^{\frac{1}{2}}=\left(e^{\log_e\frac{(x+1)^2}{(x-1)^2}}\right)^{\frac{1}{2}}$$
$$=\left(e^{\log_e\left(\frac{x+1}{x-1}\right)^2}\right)^{\frac{1}{2}}=\left(e^{2\log_e\left|\frac{x+1}{x-1}\right|}\right)^{\frac{1}{2}}=e^{\log_e\left|\frac{x+1}{x-1}\right|}=\left|\frac{x+1}{x-1}\right|$$

Example 4 (2008 VCAA Exam 1)
Let $f:R\to R$, $f(x)=e^{2x}-1$. (a) Find the rule and domain of the inverse function f^{-1}. (b) Sketch the graph of $y=f(f^{-1}(x))$ for its maximal domain. (c) Find $f(-f^{-1}(2x))$ in the form $\frac{ax}{bx+c}$ where *a*, *b* and *c* are real constants.

Solving exponential equations algebraically

Example 5 Solve the following equations.
(a) $4^x=8$ (b) $10^{2x-3}=0.01$ (c) $4^{x+1}=10^{x-1}$

(a) $4^x=8$, $(2^2)^x=2^3$, $2^{2x}=2^3$, .: $2x=3$, $x=\frac{3}{2}$.

(b) $10^{2x-3}=0.01$, $10^{2x-3}=10^{-2}$, .: $2x-3=-2$, $x=\frac{1}{2}$.

(c) $4^{x+1}=10^{x-1}$, 4 and 10 cannot be changed to the same base. Take the log (to base 10 is simpler than to base *e* in this case) of both sides of the equation.
$\log_{10}4^{x+1}=\log_{10}10^{x-1}$, $(x+1)\log_{10}4=x-1$,
$(x+1)0.6021\approx x-1$, $0.6021x+0.6021\approx x-1$, $x\approx 4.0259$.

Example 6 Solve $5e^{3x+2}=6$.

$5e^{3x+2}=6$, $e^{3x+2}=1.2$. The equivalent form is $3x+2=\log_e 1.2$

$3x=\log_e 1.2-2$, .: $x=\frac{1}{3}(\log_e 1.2-2)$ or $x\approx -0.6059$

Example 7 Solve $8e^{x+2}=3e^{2-x}$.

$8e^{x+2}=3e^{2-x}$, $\frac{e^{x=2}}{e^{2-x}}=\frac{3}{8}$, $e^{x-2-(2-x)}=\frac{3}{8}$, $e^{2x}=\frac{3}{8}$

The equivalent form is $2x=\log_e\left(\frac{3}{8}\right)$, .: $x=\frac{1}{2}\log_e\left(\frac{3}{8}\right)$ in exact form or $x\approx -0.4904$.

Alternative method: $8e^{x+2}=3e^{2-x}$, $8e^{x+2}-3e^{2-x}=0$,
$e^{2-x}(8e^{2x}-3)=0$. Since $e^{2-x}\neq 0$, .: $8e^{2x}-3=0$, $e^{2x}=\frac{3}{8}$ etc.

Example 8 Solve $3e^{2x}-4e^x+1=0$.

$3(e^x)^2-4(e^x)+1=0$,
$(3e^x-1)(e^x-1)=0$, [factorise by trial and error]
.: $3e^x-1=0$ or $e^x-1=0$
.: $e^x=\frac{1}{3}$ or $e^x=1$, .: $x=\log_e\left(\frac{1}{3}\right)$ or $x=0$.

Example 9 Solve $3e^{2x}+4e^x+1=0$.

This equation has no real solutions for *x*, because all three terms are greater than 0, .: sum > 0.

Example 10 Solve $e^{2x}-4e^x-4=0$.

$(e^x)^2-4(e^x)-4=0$ cannot be factorised over *Q*, set of rational numbers, .: use the quadratic formula to obtain
$e^x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-4)}}{2(1)}$, $e^x=2\pm 2\sqrt{2}$.
Since $e^x>0$, .: $e^x=2+2\sqrt{2}$.
Hence $x=\log_e(2+2\sqrt{2})$ in exact form or $x\approx 1.5745$.

Example 11 (2007 VCAA Exam 2)
Solve $e^{4x}-5e^{2x}+4=0$ for $x\in R$.

$e^{4x}-5e^{2x}+4=0$, $(e^{2x})^2-5e^{2x}+4=0$, $(e^{2x}-4)(e^{2x}-1)=0$,
.: $e^{2x}=1$ or $e^{2x}=4$, .: $2x=0$ or $2x=\log_e 4=2\log_e 2$,
.: $x=0$ or $\log_e 2$

Example 12 Find *k* such that the curve $y=3e^{-kx+2}$ passes through $(-2,3)$.

When $x=-2$, $y=3$, .: $3=3e^{2k+2}$, $e^{2k+2}=1$, $2k+2=0$, $k=-1$.

Example 13 Find *a* and *b* such that the curve $y=ae^{bx}+1$ passes through the points $(-1,2)$ and $(1,4)$.

Use the two points to set up two simultaneous equations:
$(-1,2)\to ae^{-b}+1=2$, $\therefore ae^{-b}=1$………(1)
$(1,4)\to ae^{b}+1=4$, $\therefore ae^{b}=3$ ………….(2)

Eq(2)/eq(1), $\frac{ae^b}{ae^{-b}}=\frac{3}{1}$, $e^{2b}=3$, $2b=\log_e 3$,

.: $b=\frac{1}{2}\log_e 3=\log_e\sqrt{3}$ ……(3)

Substitute eq(3) in eq(2), $ae^{\log_e\sqrt{3}}=3$, $a\sqrt{3}=3$, $a=\frac{3}{\sqrt{3}}=\sqrt{3}$.

Example 14 Find b and c such that the curve $y = 2e^{bx} + c$ passes through the points $(2,6)$ and $(4,10)$.

$(2,6) \to 2e^{2b} + c = 6$, .: $2e^{2b} = 6 - c$ ………..(1)

$(4,10) \to 2e^{4b} + c = 10$, .: $2e^{4b} = 10 - c$ ……..(2)

Eq(2)/eq(1), $\frac{2e^{4b}}{2e^{2b}} = \frac{10-c}{6-c}$, .: $e^{2b} = \frac{10-c}{6-c}$ ………(3)

Substitute eq(3) in eq(1), $2\left(\frac{10-c}{6-c}\right) = 6 - c$,

.: $2(10-c) = (6-c)^2$, $(6-c)^2 - 2(10-c) = 0$,

$(6-c)^2 - 2(6-c) - 8 = 0$.

Factorise to obtain $[(6-c)-4][(6-c)+2] = 0$.

.: $6 - c - 4 = 0$, i.e. $c = 2$, or $6 - c + 2 = 0$, i.e. $c = 8$.

The second result is not possible because it leads to an impossibility $2e^{2b} = -2$. .: $c = 2$ ……(4)

Substitute eq.(4) in eq.(1), $2e^{2b} = 4$, $e^{2b} = 2$, $2b = \log_e 2$,

$b = \frac{1}{2}\log_e 2 = \log_e \sqrt{2}$.

Example 15 (2006 VCAA Exam 2) Find the value of k, where $k \in R^+$, for which the equation $3 - ke^x - e^{-x} = 0$ has one or more solutions for x.

$-ke^x + 3 - e^{-x} = 0$, $e^{-x}\left(-k(e^x)^2 + 3e^x - 1\right) = 0$

Since $e^{-x} \neq 0$, $-k(e^x)^2 + 3e^x - 1 = 0$. This is a quadratic equation in e^x and e^x is a one to one function. For it to have one or more solutions, $\Delta \geq 0$, i.e. $3^2 - 4(-k)(-1) \geq 0$,

$9 - 4k \geq 0$, $9 \geq 4k$, $k \leq \frac{9}{4}$

Solving logarithmic equations algebraically

Always check each solution by substituting it into the original equation to see whether each log is defined.

Example 16 Solve the following equations for x.

(a) $\log_2 x = 5$ (b) $\log_2(x^2 - 1) = 3$ (c) $\log_2\left(\frac{1}{x}\right) = 1.5$

(a) $\log_2 x = 5$, the equivalent form is $x = 2^5$, $x = 32$.

(b) $\log_2(x^2 - 1) = 3$, the equivalent form is $x^2 - 1 = 2^3$, $x^2 = 9$, $x = \pm 3$.

(c) $\log_2\left(\frac{1}{x}\right) = 1.5$, the equivalent form is $\frac{1}{x} = 2^{1.5}$, $x = 2^{-1.5}$

Example 17 (2013 VCAA Exam 1)
Solve $2\log_3(5) - \log_3(2) + \log_3(x) = 2$ for x.

$\log_3(25) - \log_3(2) + \log_3(x) = 2$, $\log_3\left(\frac{25x}{2}\right) = 2$,

$\frac{25x}{2} = 3^2$, $x = \frac{18}{25}$

Example 18 Solve $\log_x \sqrt{243} = 2.5$ for x, where $x > 0$.

$\log_x \sqrt{243} = 2.5$, the equivalent form is $\sqrt{243} = x^{2.5}$

$\sqrt{3^5} = x^{2.5}$, $(3^5)^{\frac{1}{2}} = x^{2.5}$, $3^{2.5} = x^{2.5}$, .: $x = 3$

Example 19 Solve $\log_x 5x = 3$ for x, where $x > 0$.

$\log_x 5x = 3$, $\log_x 5 + \log_x x = 3$, $\log_x 5 + 1 = 3$, $\log_x 5 = 2$, the equivalent form is $x^2 = 5$, .: $x = \sqrt{5}$

Example 20 Solve $\log_{10}(x+1) - \log_{10} x = 1$ for x.

$\log_{10}(x+1) - \log_{10} x = 1$, $\log_{10}\left(\frac{x+1}{x}\right) = 1$, the equivalent form is $\frac{x+1}{x} = 10^1$, $x + 1 = 10x$, $1 = 9x$, $x = \frac{1}{9}$.

Example 21 Solve $\log_{10}(\sqrt{35} + x) + \log_{10}(\sqrt{35} - x) = 1$.

$\log_{10}(\sqrt{35} + x) + \log_{10}(\sqrt{35} - x) = 1$,

$\log_{10}(\sqrt{35} + x)(\sqrt{35} - x) = 1$,

the equivalent form is $(\sqrt{35} + x)(\sqrt{35} - x) = 10^1$,

.: $35 - x^2 = 10$, $x^2 = 25$, $x = \pm 5$.

Example 22 Solve $2\log_e(x-2) - \log_e(x+1) = 0$.

$2\log_e(x-2) - \log_e(x+1) = 0$, $\log_e \frac{(x-2)^2}{x+1} = 0$,

$\frac{(x-2)^2}{x+1} = 1$, $(x-2)^2 = x + 1$, $x^2 - 4x + 4 = x + 1$,

$x^2 - 5x + 3 = 0$, .: $x = \frac{5 \pm \sqrt{25 - 12}}{2}$,

i.e. $x = \frac{5 + \sqrt{13}}{2}$ or $\frac{5 - \sqrt{13}}{2}$. Only the first solution is correct because $x > 2$ for $2\log_e(x-2)$ to be defined.

Example 23 $(-2,1)$ is a point on the curve $y = 2\log_e(1 - ax)$. Find a.

$(-2,1) \to 1 = 2\log_e(1 - a(-2))$, $\frac{1}{2} = \log_e(1 + 2a)$,

$1 + 2a = e^{\frac{1}{2}}$, $a = \frac{1}{2}(\sqrt{e} - 1)$.

Example 24 Find a and b such that $y = \log_e \sqrt{ax + b} + 1$ passes through the points $(0,1)$ and $\left(1, \frac{3}{2}\right)$.

$(0,1) \to 1 = \log_e \sqrt{b} + 1$, $\log_e \sqrt{b} = 0$, $\sqrt{b} = 1$, $b = 1$……(1)

$\left(1, \frac{3}{2}\right) \to \frac{3}{2} = \log_e \sqrt{a+b} + 1$, $\log_e \sqrt{a+b} = \frac{1}{2}$, $\sqrt{a+b} = e^{\frac{1}{2}}$,

$a + b = e$ ……(2)

Substitute eq(1) in eq(2), $a = e - 1$.

Example 25 (2007 VCAA Exam 1) Solve the equation $\log_e(3x+5) + \log_e 2 = 2$ for x.

$\log_e(3x+5) + \log_e 2 = 2$, $\log_e 2(3x+5) = 2$, $2(3x+5) = e^2$,

$3x + 5 = \frac{e^2}{2}$, $3x = \frac{e^2}{2} - 5$, $x = \frac{1}{3}\left(\frac{e^2}{2} - 5\right)$

Example 26 Find the exact coordinates of the intersection of the graphs of $y = \log_{10}(x-1)$ and $y = \log_{10} x - 1$.

Let $\log_{10} x - 1 = \log_{10}(x-1)$.

$\log_{10} x - \log_{10}(x-1) = 1$, $\log_{10}\left(\frac{x}{x-1}\right) = 1$, .: $\frac{x}{x-1} = 10^1$

$x = 10x - 10$, $x = \frac{10}{9}$, .: $y = \log_{10} x - 1 = \log_{10}\frac{10}{9} - 1 = -\log_{10} 9$

The graphs intersect at $\left(\frac{10}{9}, -\log_{10} 9\right)$.

Exercise: Next page

Q1 Simplify (a) $e^{2\log_e a}$ (b) $\log_e\left(2e^{-a}\right)$ (c) $\sqrt[5]{32(x+1)^{10}}$

Q2 Solve the following equations.
(a) $4^x = 32$ (b) $10^{2x-3} = 0.0001$ (c) $4^{x+1} = 100^{x-1}$.

Q3 Solve $2e^{5x-2} = 10$.

Q4 Solve $3e^{2x+1} = 5e^{1-x}$.

Q5 Solve $e^{4x} - 7e^{2x} + 12 = 0$ for $x \in R$.

Q6 Find a and b such that the curve $y = ae^{bx} + 1$ passes through the points $(1,2)$ and $(0, e+1)$.

Q7 Find b and c such that the curve $y = e^{bx} + c$ passes through the points $(0,2)$ and $\left(1, e^2 + 1\right)$.

Q8 Find the value of k for which the equation $2 + ke^x - 3e^{-x} = 0$ has one or more solutions for x.

Q9 Solve the following equations for x.
(a) $\log_5 x = 2$ (b) $\log_{10}\left(x^2 + 19\right) = 2$ (c) $\log_e\left(\frac{1}{x}\right) = e$

Q10 Solve $\log_x 2x = 2$ for x, where $x > 0$.

Q11 Solve $\log_{10}(x+1) + \log_{10} x = 1$ for x.

Q12 Solve $\log_e(x+2) - \frac{1}{2}\log_e(1-x) = 0$.

Numerical, algebraic and worded answers: 1a. a^2 1b. $\log_e 2 - a$ 1c. $2(x+1)^2$ 2a. $x = 5/2$ 2b. $x = -1/2$ 2c. $x = (1 + \log_{10} 2)/(1 - \log_{10} 2)$ 3. $x = (2 + \log_e 5)/5$ 4. $x = (1/3)\log_e(5/3)$ 5. $x = \log_e 2$ or $(\log_e 3)/2$ 6. $a = e$, $b = -1$ 7. $b = 2$, $c = 1$ 8. $k \geq 1/3$ 9a. $x = 25$ 9b. $x = \pm 9$ 9c. $x = e^{-e}$ 10. $x = 2$ 11. $x = \left(\sqrt{41} - 1\right)/2$ 12. $x = \left(3 - \sqrt{13}\right)/2$