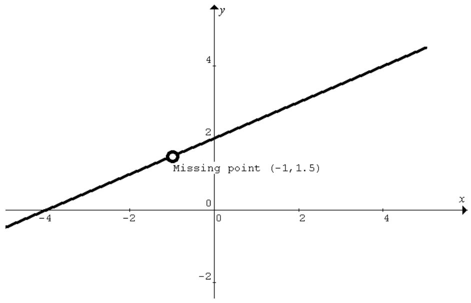
Calculus

Limits and continuity

Sometimes the value of a function $f(x)$ *at* $x=\alpha$ *may not be defined, but* $f(x)$ *may get close to certain value L as x gets close to* α *from both sides of* α. *We say L is the limit of* $f(x)$ *as x approaches* α. *This idea is expressed as* $\lim_{x\to\alpha} f(x)=L$ *or* $\lim_{h\to 0} f(\alpha+h)=L$, *where* $h=\Delta x$.

If x approaches α *from the left,* $x\to\alpha^-$, *h is a negative value.*
If x approaches α *from the right,* $x\to\alpha^+$, *h is a positive value.*

Example 1

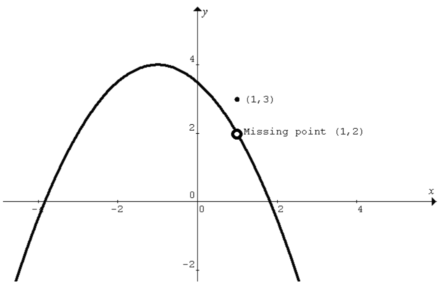


The above function $f(x)$ is undefined at $x=-1$, i.e. $f(-1)$ does not exist. However, as $x\to -1$ from either side of $x=-1$, $f(x)\to 1.5$, .: $\lim_{x\to -1} f(x)=1.5$ or $\lim_{h\to 0} f(-1+h)=1.5$

We say the limit of $f(x)$ exists as *x* approaches -1 and it is 1.5.

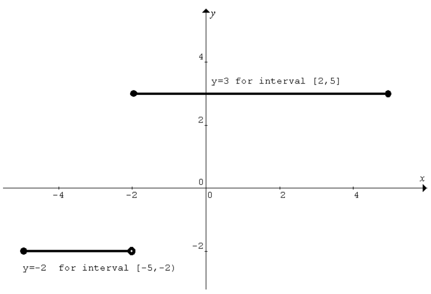
The function is said to be **discontinuous** at $x=-1$ because $f(-1)$ does not exist.

Example 2



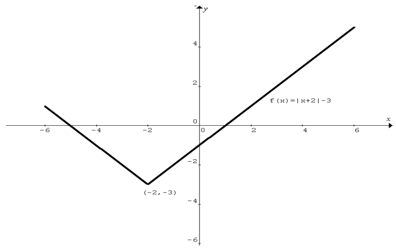
The function $f(x)$ is defined at $x=1$, and $f(1)=3$. However, as $x\to 1$ from either side of $x=1$, $f(x)\to 2$, .: $\lim_{x\to 1} f(x)=2$ or $\lim_{h\to 0} f(1+h)=2$, i.e. the limit of $f(x)$ exists as *x* approaches 1 and it is 2. The function is discontinuous at $x=1$ because $\lim_{x\to 1} f(x)\neq f(1)$

Example 3



The above function $f(x)$ is defined at $x=-2$, and $f(-2)=3$. As $x\to -2$ from the left side of $x=-2$, $f(x)\to -2$, and as $x\to -2$ from the right side, $f(x)\to 3$. The left and right limits are different, .: $\lim_{x\to -2} f(x)$ or $\lim_{h\to 0} f(-2+h)$ does not exist. The function is said to be discontinuous at $x=-2$ because the left and right limits are different, i.e. $\lim_{x\to -2} f(x)$ does not exist .

Example 4



The limit of $f(x)$ exists as $x\to -2$, i.e. $\lim_{x\to -2} f(x)=-3$. The value of the function at $x=-2$ is defined, i.e. $f(-2)=-3$.

Since $\lim_{x\to -2} f(x)=f(-2)$, .: $f(x)$ is **continuous** at $x=-2$.

In general, a function $f(x)$ is continuous at $x=\alpha$ if $\lim_{x\to\alpha} f(x)$ and $f(\alpha)$ both exist and $\lim_{x\to\alpha} f(x)=f(\alpha)$.

Differentiability of a function at a point on an interval

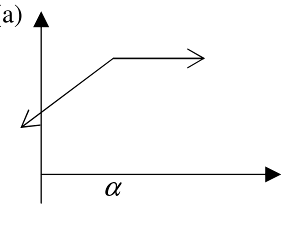
A function $f(x)$ is differentiable at $x=\alpha$ if it is continuous and there is no abrupt change in its gradient at $x=\alpha$, i.e. the section on the left of $x=\alpha$ is *smoothly* joined to the section on the right, and the curve appears to be a straight section (**local linearity**) in the immediate neighbourhood of $x=\alpha$.

A function $f(x)$ is differentiable on a closed interval $[p,q]$ if $f(x)$ is differentiable at each point of the open interval (p,q) AND $\lim_{x\to p^+}\frac{f(x)-f(p)}{x-p}$ and $\lim_{x\to q^-}\frac{f(x)-f(q)}{x-q}$ both exist.

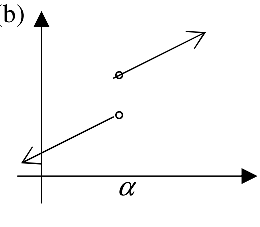
A function is **not** differentiable at an **open** end point.

Example 5 Discuss the differentiability of the following functions at $x=\alpha$.

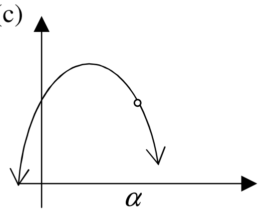
(a)



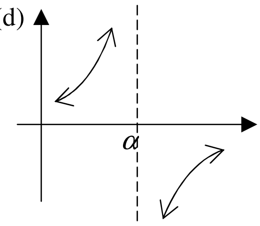
(b)



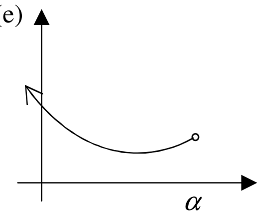
(c)



(d)

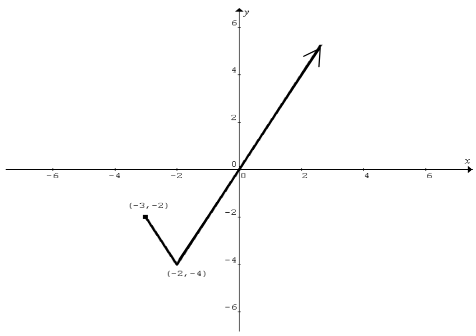


(e)



(a) The function has an abrupt change in its gradient at $x=\alpha$, .: it is not differentiable at $x=\alpha$.
(b) The function is not continuous at $x=\alpha$, .: it is not differentiable at $x=\alpha$.
(c) The function is not continuous at $x=\alpha$, .: it is not differentiable at $x=\alpha$.
(d) The function is undefined at $x=\alpha$, .: it is not differentiable at $x=\alpha$.
(e) It is an open end point of the function at $x=\alpha$, .: it is not differentiable at $x=\alpha$.

Example 6 Discuss the differentiability of the following modulus function $f(x)$ on the unbounded interval $[-3,\infty)$.



$f(x)$ is not differentiable at the vertex $(-2,-4)$, but it is differentiable on the interval $[-3,-2)$ or $(-2,\infty)$.

Graph of $f'(x)$ from graph of $f(x)$

$f'(x)$ denotes the gradient function of function $f(x)$. $f'(x)$ is undefined at points where $f(x)$ is not differentiable.

Steps in sketching the graph of $f'(x)$ from the graph of $f(x)$:

(1) Look for stationary points where $f'(x)=0$. They are the x-intercepts of $f'(x)$ graph.

(2) Check the gradient (+/–) on each side of a stationary point of $f(x)$. The $f'(x)$ graph is above the x-axis for +, below the x-axis for –. If the two sides have opposite signs, the $f'(x)$ graph cuts across the x-axis. If the two sides have the same sign, the x-intercept of $f'(x)$ graph is a turning point.

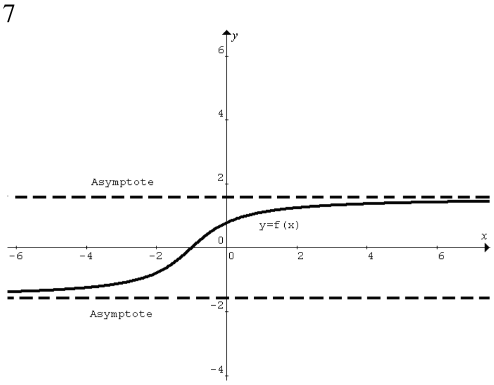
(3) $f'(x)$ has the same vertical asymptotes as $f(x)$.

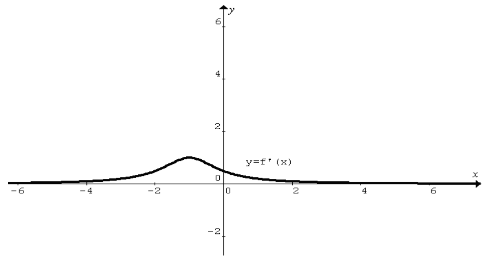
(4) The horizontal asymptote(s) of $f(x)$ corresponds to the asymptote $y=0$ (i.e. the x-axis) of $f'(x)$.

(5) An oblique asymptote of $f(x)$ corresponds to a horizontal asymptote of $f'(x)$.

The graph of $f'(x)$ remains the same if $f(x)$ is vertically translated.

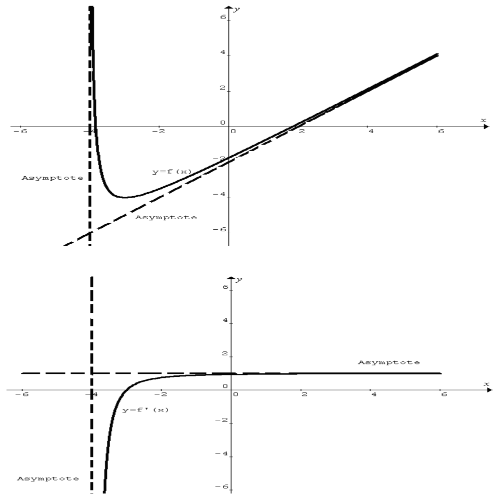
Example 7





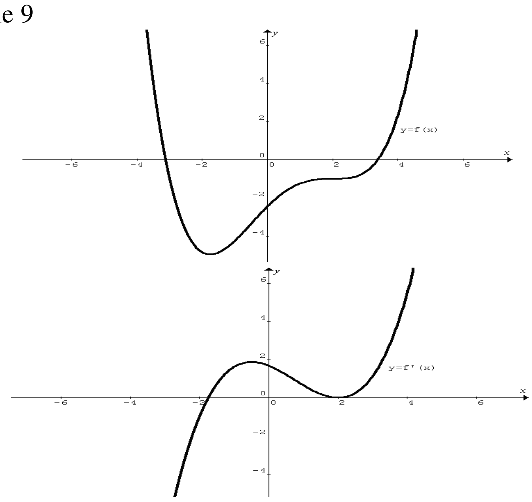
Domain of $f(x)$: R, domain of $f'(x)$: R.

Example 8



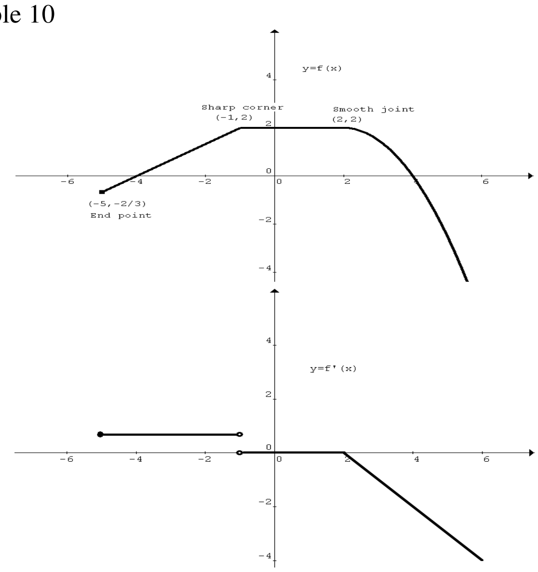
Domain of $f(x)$: $(-4,\infty)$, domain of $f'(x)$: $(-4,\infty)$.

Example 9



Domain of $f(x)$: R, domain of $f'(x)$: R.

Example 10



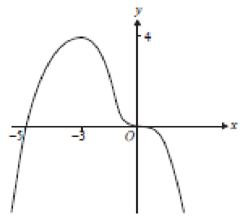
Dom. of $f(x)$: $[-5,\infty)$, dom. of $f'(x)$: $[-5,-1)\cup(-1,\infty)$.

Exercise: Next page

Q1 (2011 VCAA Exam 2)

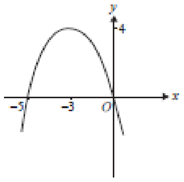
Question 9

The graph of the function $y = f(x)$ is shown below.

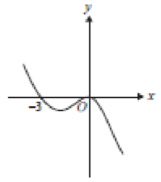


Which of the following could be the graph of the derivative function $y = f'(x)$?

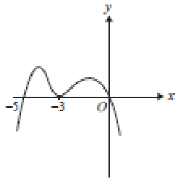
A.



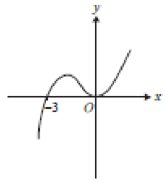
B.



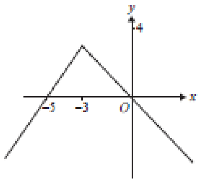
C.



D.



E.



Q2 (2010 VCAA Exam 2)

Question 17

The function f is differentiable for all $x \in R$ and satisfies the following conditions.

- $f'(x) < 0$ where $x < 2$
- $f'(x) = 0$ where $x = 2$
- $f'(x) = 0$ where $x = 4$
- $f'(x) > 0$ where $2 < x < 4$
- $f'(x) > 0$ where $x > 4$

Which one of the following is true?

A. The graph of f has a local maximum point where $x = 4$.
B. The graph of f has a stationary point of inflection where $x = 4$.
C. The graph of f has a local maximum point where $x = 2$.
D. The graph of f has a local minimum point where $x = 4$.
E. The graph of f has a stationary point of inflection where $x = 2$.

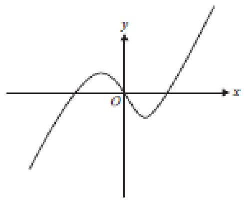
Question 18

For the function $f(x) = e^{|x|} - 1$, which of the following statements is true?

A. The function is increasing for all x.
B. The function has an asymptote at $y = -1$.
C. The function is not continuous at $x = 0$.
D. The function is not differentiable at $x = 0$.
E. The function has a stationary point at $x = 0$.

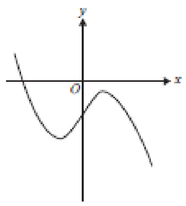
Question 19

The graph of the gradient function $y = f'(x)$ is shown below.

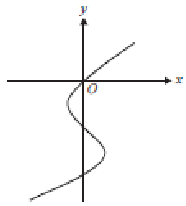


Which of the following could represent the graph of the function $f(x)$?

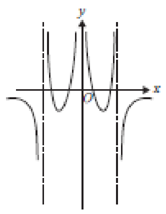
A.



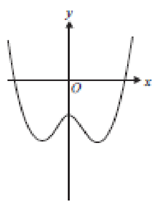
B.



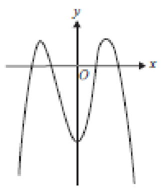
C.



D.



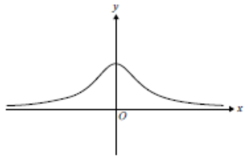
E.



Q3 (2008 VCAA Exam 2)

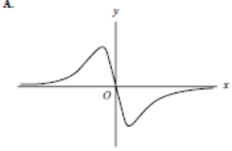
Question 19

The graph of a function f is shown below.

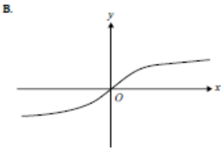


The graph of an antiderivative of f could be

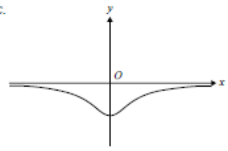
A.



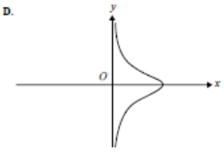
B.



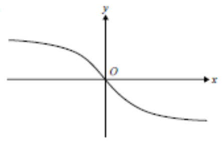
C.



D.

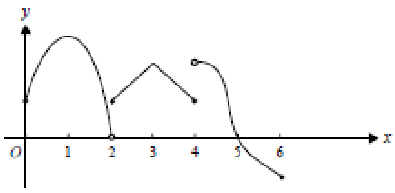


E.



Question 22

The graph of the function f with domain $[0, 6]$ is shown below.



Which one of the following is **not** true?

A. The function is not continuous at $x = 2$ and $x = 4$.
B. The function exists for all values of x between 0 and 6.
C. $f(x) = 0$ for $x = 2$ and $x = 5$.
D. The function is positive for $x \in [0, 5)$.
E. The gradient of the function is not defined at $x = 4$.

Q4 (2007 VCAA Exam 2)

Question 8

Which one of the following is **not** true about the function $f: R \to R$, $f(x) = |2x + 4|$?

A. The graph of f is continuous everywhere.
B. The graph of f' is continuous everywhere.
C. $f(x) \geq 0$ for all values of x
D. $f'(x) = 2$ for all $x > 0$
E. $f'(x) = -2$ for all $x < -2$

Question 12

Let $f: R \to R$ be a differentiable function such that

- $f'(3) = 0$
- $f'(x) < 0$ when $x < 3$ and when $x > 3$

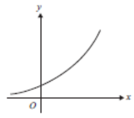
When $x = 3$, the graph of f has a

A. local minimum.
B. local maximum.
C. stationary point of inflection.
D. point of discontinuity.
E. gradient of 3.

Q5 (2006 VCAA Exam 2 Sample Version 2)

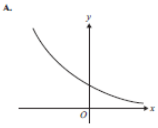
Question 18

The graph of the function f, with rule $y = f(x)$, is shown below.

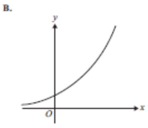


Which one of the following could be the graph of the curve with equation $y = f'(x)$?

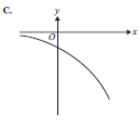
A.



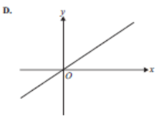
B.



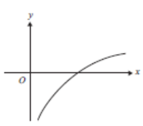
C.



D.



E.



Answers:
Q1 B
Q2 B, D, D
Q3 B, C
Q4 B, C
Q5 B