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Free maths self-help revision

Rules for finding the derivatives of power functions x^n for $n \in Q$

The general rules are: $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$,

$$\frac{d}{dx}\left(ax^n\right) = nax^{n-1},$$

$$\frac{d}{dx}\left((x-b)^n\right) = n(x-b)^{n-1},$$

$$\frac{d}{dx}\left(a(kx-b)^n\right) = kna(kx-b)^{n-1}.$$

Try to remember the derivative of $x^{\frac{1}{2}}$, or $\sqrt{x}$:

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}},$$

Example 1 Find the derivative of $\frac{2\left(x+\frac{1}{2}\right)^{\frac{3}{2}}}{3}$.

$$\frac{d}{dx}\left(\frac{2}{3}\left(x+\frac{1}{2}\right)^{\frac{3}{2}}\right) = \frac{3}{2}\times\frac{2\left(x+\frac{1}{2}\right)^{\frac{1}{2}}}{3} = \left(x+\frac{1}{2}\right)^{\frac{1}{2}}.$$

Example 2 Given $y = 2\sqrt{3x-5}$, find $\frac{dy}{dx}$ at $x = 3$.

$$\frac{dy}{dx} = \frac{2\times 3}{2\sqrt{3x-5}} = \frac{3}{\sqrt{3x-5}} = \frac{3}{\sqrt{3(3)-5}} = \frac{3}{2}.$$

Example 3 Find the gradient function of $f(x) = \frac{5}{\sqrt{x}} - \frac{1}{2x+3}$.

$$f(x) = \frac{5}{\sqrt{x}} - \frac{1}{2x+3} = 5x^{-\frac{1}{2}} - (2x+3)^{-1}.$$

$$f'(x) = -\frac{1}{2}\times 5x^{-\frac{1}{2}-1} - {}^{-}1\times 2(2x+3)^{-1-1} = -\frac{5}{2}x^{-\frac{3}{2}} + 2(2x+3)^{-2}$$

or $-\frac{5}{2x^{\frac{3}{2}}} + \frac{2}{(2x+3)^2}$.

Example 4 Differentiate $\frac{3x^2+5x-1}{x^2}$.

$$\frac{3x^2+5x-1}{x^2} = \frac{3x^2}{x^2} + \frac{5x}{x^2} - \frac{1}{x^2} = 3 + 5x^{-1} - x^{-2}.$$

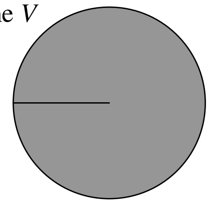
$$\frac{d}{dx}\left(3+5x^{-1}-x^{-2}\right) = 0 + {}^{-}1\times 5x^{-2} - {}^{-}2x^{-3} = -5x^{-2} + 2x^{-3}$$

$$= -\frac{5}{x^2} + \frac{2}{x^3} \text{ or } \frac{-5x+2}{x^3}.$$

Example 5 Find the gradient of the tangent to the curve $y = \frac{5(1-3x)^2}{2}$ at $x = \frac{4}{15}$.

Gradient $= \frac{dy}{dx} = {}^{-}3\times 2\times\frac{5(1-3x)}{2} = {}^{-}15(1-3x) = -3$.

Example 6 Find the rate of change of the volume of a sphere with respect to its radius when the radius is 2 units.



Volume of a sphere $V(r) = \frac{4}{3}\pi r^3$.

Required rate $= \frac{dV}{dr} = 4\pi r^2 = 4\pi(2)^2 = 16\pi$ sq. units.

Rules for finding the derivatives of e^x and $\log_e x$

The general rules are: $\frac{d}{dx}\left(e^x\right) = e^x$,

$$\frac{d}{dx}\left(ae^{kx}\right) = kae^{x},$$

$$\frac{d}{dx}\left(e^{x-b}\right) = e^{x-b},$$

$$\frac{d}{dx}\left(ae^{kx-b}\right) = kae^{kx-b}.$$

$$\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x},$$

$$\frac{d}{dx}\left(a\log_e(kx)\right) = \frac{a}{x},$$

$$\frac{d}{dx}\left(\log_e(x-b)\right) = \frac{1}{x-b},$$

$$\frac{d}{dx}\left(a\log_e(kx-b)\right) = \frac{ka}{kx-b},$$

$$\frac{d}{dx}\left(a\log_e k(x-b)\right) = \frac{a}{x-b}.$$

Note: The above rules are for exponential and logarithmic functions with base e only. See examples below for other bases.

Example 7 Find the derivative of $3e^{1-2x}$.

$$\frac{d}{dx}\left(3e^{1-2x}\right) = {}^{-}2\times 3e^{1-2x} = -6e^{1-2x}.$$

Example 8 Find the gradient of $f(x) = 1 - 2\log_e 3(x+1)$ at $x = 0$.

$f'(x) = -\frac{2}{x+1}$. Gradient $= f'(0) = -2$ at $x = 0$.

Example 9 Given $x = \frac{e^{2t+1}-e^{1-2t}}{e^{1-t}}$, find the rate of change of x with respect to t.

$$x = \frac{e^{2t+1}-e^{1-2t}}{e^{1-t}} = e^{2t+1-(1-t)} - e^{1-2t-(1-t)} = e^{3t} - e^{-t}$$

$$\frac{dx}{dt} = 3e^{3t} + e^{-t}$$

Example 10 Differentiate $\log_e\frac{(2x+1)(x-1)}{x^2-4}$.

$$\log_e\frac{(2x+1)(x-1)}{x^2-4} = \log_e\frac{(2x+1)(x-1)}{(x-2)(x+2)}$$

$$= \log_e(2x+1) + \log_e(x-1) - \log_e(x-2) - \log_e(x+2).$$

$$\frac{d}{dx}\left(\log_e\frac{(2x+1)(x-1)}{x^2-4}\right) = \frac{2}{2x+1} + \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x+2}.$$

Example 11 Find the gradient function of $f(x)=3\log_{10}(x+1)$.

Change to base *e*: $f(x)=\dfrac{3\log_e(x+1)}{\log_e 10}$, $f'(x)=\dfrac{3}{(x+1)\log_e 10}$.

Example 12 Find the gradient of the tangent to the curve $y=a\times 2^{\frac{x}{3}}$ at $x=3$ in terms of *a*.

Change to base *e*: $y=a\times e^{\log_e 2^{\frac{x}{3}}}=ae^{\frac{x}{3}\log_e 2}=ae^{\frac{\log_e 2}{3}x}$.

$$\frac{dy}{dx}=\frac{\log_e 2}{3}\left(ae^{\frac{\log_e 2}{3}x}\right)=\frac{\log_e 2}{3}\left(a\times 2^{\frac{x}{3}}\right).$$

At $x=3$, gradient $=\dfrac{2a\log_e 2}{3}$.

Rules for finding the derivatives of $\sin(x)$, $\cos(x)$, $\tan(x)$

The general rules are: $\dfrac{d}{dx}(\sin(x))=\cos(x)$

$$\frac{d}{dx}(a\sin(kx))=ka\cos(kx)$$
$$\frac{d}{dx}(\sin(x-b))=\cos(x-b)$$
$$\frac{d}{dx}(a\sin(kx-b))=ka\cos(kx-b)$$
$$\frac{d}{dx}(a\sin k(x-b))=ka\cos k(x-b)$$

$$\frac{d}{dx}(\cos(x))=-\sin(x)$$
$$\frac{d}{dx}(a\cos(kx))=-ka\sin(kx)$$
$$\frac{d}{dx}(\cos(x-b))=-\sin(x-b)$$
$$\frac{d}{dx}(a\cos(kx-b))=-ka\sin(kx-b)$$
$$\frac{d}{dx}(a\cos k(x-b))=-ka\sin k(x-b)$$

$$\frac{d}{dx}(\tan(x))=\sec^2(x),\ \left[\sec(x)=\frac{1}{\cos(x)}\right]$$
$$\frac{d}{dx}(a\tan(kx))=ka\sec^2(kx)$$
$$\frac{d}{dx}(\tan(x-b))=\sec^2(x-b)$$
$$\frac{d}{dx}(a\tan(kx-b))=ka\sec^2(kx-b)$$
$$\frac{d}{dx}(a\tan k(x-b))=ka\sec^2 k(x-b)$$

Example 13 Given $y=-\dfrac{2}{3}\cos 3\pi(x+5)$, find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx}={}^{-}3\pi\times\frac{{}^{-}2}{3}\sin 3\pi(x+5)=2\pi\sin 3\pi(x+5).$$

Example 14 Find $\dfrac{d}{dx}\left(\dfrac{3\sin\left(2x-\frac{\pi}{3}\right)}{5}\right)$.

$$\frac{d}{dx}\left(\frac{3\sin\left(2x-\frac{\pi}{3}\right)}{5}\right)=\frac{2\times 3\cos\left(2x-\frac{\pi}{3}\right)}{5}=\frac{6}{5}\cos\left(2x-\frac{\pi}{3}\right).$$

Example 15 Find the derivative of $y=\dfrac{\sin\frac{\pi}{6}(x+1)}{2\cos\frac{\pi}{6}(x+1)}$.

Change to $y=\dfrac{1}{2}\tan\left(\dfrac{\pi}{6}(x+1)\right)$.

$$y'=\frac{\pi}{6}\times\frac{1}{2}\sec^2\left(\frac{\pi}{6}(x+1)\right)=\frac{\pi}{12}\sec^2\left(\frac{\pi}{6}(x+1)\right).$$

Example 16 Find the derivative of $y=10^{10}\sin^2(\pi x)+10^{10}\cos^2(\pi x)+10^{10}$.

Simplify, $y=10^{10}\sin^2(\pi x)+10^{10}\cos^2(\pi x)+10^{10}$

$=10^{10}\left(\sin^2(\pi x)+\cos^2(\pi x)+1\right)$

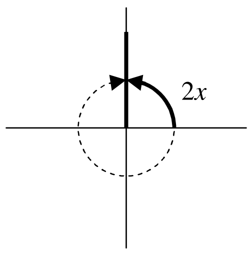
$=10^{10}(1+1)=2\times 10^{10}$. It is a constant. Hence, $y'=0$.

Example 17 Find $-\pi\le x\le\pi$ where the gradient of the curve $y=3\cos(2x)-1$ is –6.

Gradient $=\dfrac{dy}{dx}=2\times{}^{-}3\sin(2x)=-6\sin(2x)$.

Let $\dfrac{dy}{dx}=-6$. $-6\sin(2x)=-6$, $\sin(2x)=1$.

Solve $\sin(2x)=1$ for *x*, where $-\pi\le x\le\pi$, i.e. $-2\pi\le 2x\le 2\pi$.



$2x=-\dfrac{3\pi}{2}$ or $\dfrac{\pi}{2}$, $\therefore x=-\dfrac{3\pi}{4}$ or $\dfrac{\pi}{4}$.

Rules for finding the derivatives of modulus functions

Since $|x|=\sqrt{x^2}$, $\dfrac{d}{dx}|x|=\dfrac{d}{dx}\sqrt{x^2}=\dfrac{2x}{2\sqrt{x^2}}=\dfrac{x}{|x|}=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$

The general rules are: $\dfrac{d}{dx}|x|=\dfrac{x}{|x|}$

$$\frac{d}{dx}(a|kx|)=\frac{ak(kx)}{|kx|}$$
$$\frac{d}{dx}|x-b|=\frac{x-b}{|x-b|}$$
$$\frac{d}{dx}|kx-b|=\frac{k(kx-b)}{|kx-b|}$$
$$\frac{d}{dx}|k(x-b)|=\frac{k^2(x-b)}{|k(x-b)|}$$

Example 18 Find the derivative of $\left|\dfrac{2x+1}{5}\right|$.

$$\left|\frac{2x+1}{5}\right|=\left|\frac{2}{5}x+\frac{1}{5}\right|$$
$$\frac{d}{dx}\left|\frac{2x+1}{5}\right|=\frac{\frac{2}{5}\left(\frac{2}{5}x+\frac{1}{5}\right)}{\left|\frac{2}{5}x+\frac{1}{5}\right|}=\frac{2(2x+1)}{5|2x+1|}.$$

Exercise: Next page

1. Differentiate $-\frac{5(1-2x)^4}{2}$ with respect to x.

2. Given $f(x) = -\frac{2}{5(1-2x)^4}$, find $f'(x)$.

3. Find $\frac{d}{dx}[(3x-2)\log_e \pi]$.

4. Find the derivative of $(1+3x)^2\sqrt{1+3x}$.

5. Evaluate $f'(-1)$, given $f(x) = \frac{1}{2}\left(\sqrt[3]{\frac{8}{1-x}}\right)$.

6. Find $\frac{d}{d(x^2)}\left(\frac{\sqrt{a+x^2}}{(a+x^2)^2}\right)$.

7. Differentiate $\frac{2e^{2(3-x)}}{3}$ with respect to x.

8. Given $y = 3\times 2^{x+1}$, find $\frac{dy}{dx}$.

9. Find $\frac{d}{dx}\log_e\left(\frac{2x-1}{3}\right)$.

10. Given $f(x) = -3\log_{10}(3x)$, find $f'(x)$.

11. Differentiate $2e^{3\log_e\sqrt{x+2}}$ with respect to x.

12. Find the derivative of $\left|\frac{1-3x}{2}\right|$.

Numerical, algebraic and worded answers: 1. $20(1-2x)^3$ 2. $-16 / 5(1-2x)^5$ 3. $3\log_e\pi$ 4. $(15/2)(1+3x)^{3/2}$ 5. $1/ 3(1-x)^{4/3}$
6. $-3 / 2(a+x^2)^{5/2}$ 7. $(-4/3)e^{2(3-x)}$ 8. $(3\log_e 2)2^{x+1}$ 9. $2 / (2x-1)$ 10. $-3 / x\log_e 10$ 11. $3\sqrt{(x+2)}$ 12. $3(1-3x)/ 2|1-3x|$