itute.com

**Free maths self-help revision** 

**Anti-differentiation**

*Standard notation for anti-differentiation of function* $f(x)$ *is* $\int f(x)dx$ . *Anti-differentiation is the reverse process of differentiation. Anti-differentiation and differentiation 'undo' each other.*

$\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$ *and* $\int\left(\frac{d}{dx}(f(x))\right)dx = f(x) + C$ , *where C is a constant. The inclusion of C in the second expression is due to the fact that* $\frac{d}{dx}(f(x)+C) = \frac{d}{dx}(f(x))$.

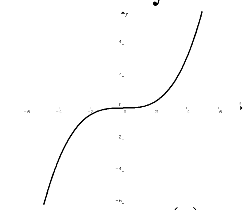
| $f(x)+C$ | $\frac{d}{dx}(f(x)+C)$ |
|---|---|
| $x^2 - 1000000$ | $2x$ |
| $x^2$ | $2x$ |
| $x^2 + 1000000$ | $2x$ |

**Relation between graph of anti-derivative function and graph of original function**

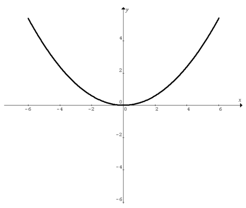
*Since* $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$, $\therefore$ *the gradient of the graph of the anti-derivative function gives the value of the original function. This is the same relationship as the graph of* $f(x)$ *and the graph of* $f'(x)$ *discussed previously).*

*There are infinitely many anti-derivative graphs that correspond to the graph of the original function.*

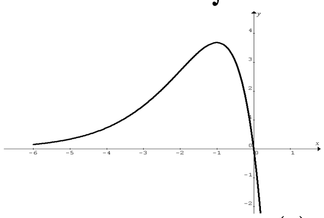
Example 1 $y = \int f(x)dx$



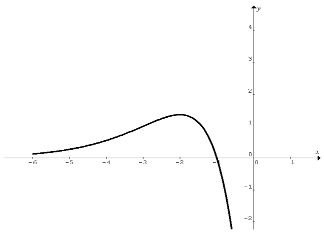
$y = f(x)$



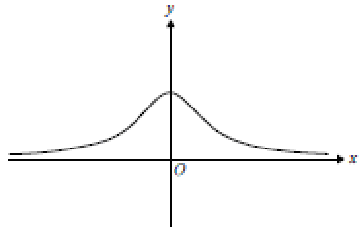
Example 2 $y = \int f(x)dx$



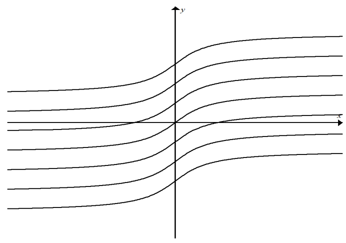
$y = f(x)$



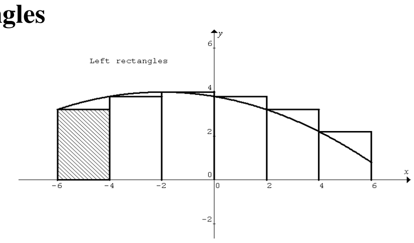
Example 3 The graph of $y = f(x)$ is shown below. Sketch a graph of $y = \int f(x)dx$ .

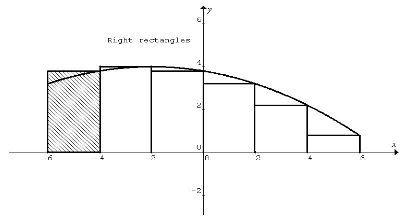


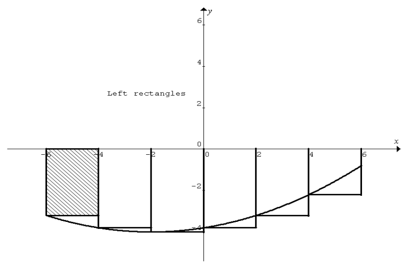
There are infinitely many anti-derivative graphs that correspond to the graph of $y = f(x)$. Take any one of the following graphs.

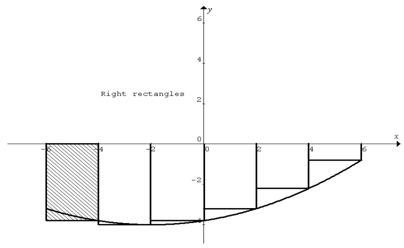


**Approximate area 'under' a curve by left rectangles and right rectangles**




Example 4 Estimate the area bounded by the curve $y = -0.05(x+2)^2 + 4$, the $x$-axis, $x = -6$ and $x = 6$ by left rectangles 2 units wide.

The first left rectangle has $y = f(-6) = 3.2$ as its height, the height of the second left rectangle $f(-4) = 3.8$, the third $f(-2) = 4$, etc.

$Area \approx 3.2\times 2 + 3.8\times 2 + 4\times 2 + 3.8\times 2 + 3.2\times 2 + 2.2\times 2$
$= (3.2 + 3.8 + 4 + 3.8 + 3.2 + 2.2)\times 2 = 40.4$ square units.

Example 5 Estimate the area bounded by the curve $y=-0.05(x+2)^2+4$, the $x$-axis, $x=-6$ and $x=6$ by right rectangles 2 units wide.

The first right rectangle has $y=f(-4)=3.8$ as its height, the height of the second right rectangle $f(-2)=4$, the third $f(0)=3.8$, etc.

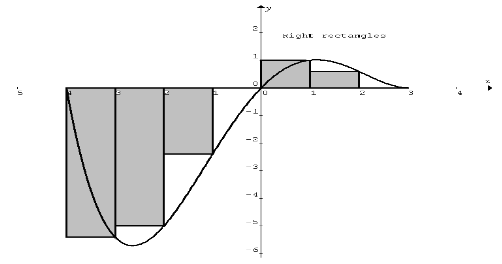
$Area \approx 3.8\times2+4\times2+3.8\times2+3.2\times2+2.2\times2+0.8\times2$
$=(3.8+4+3.8+3.2+2.2+0.8)\times2=35.6$ square units.

Note: The average of the above two results give a better estimate of the required area, i.e. $\dfrac{40.2+35.6}{2}=37.9$ square units.

Using CAS/graphics calculator, $\int f(x)dx=38.4$ square units.

Example 6 Using right rectangles of unit width, estimate the area bounded by the curve $y=0.1x(x+4)(x-3)^2$ and the $x$-axis.

The given function is in factorised form, the linear factors indicate the location of the $x$-intercepts: $x=-4$, 0, 3. The point at $x=3$ is a turning point.

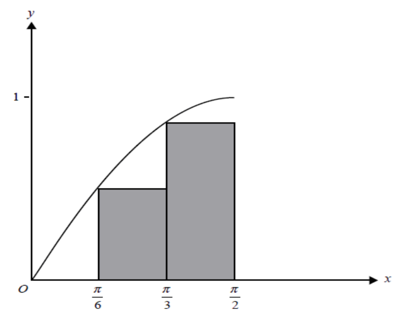



For $-4\le x\le0$, $f(x)\le0$, $\therefore$ the height of a rectangle $=|f(a)|$ for $-4\le a\le0$.

| $x$ | - 3 | - 2 | - 1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $\lvert f(x)\rvert$ | 5.4 | 5 | 2.4 | 0 | 1 | 0.6 | 0 |

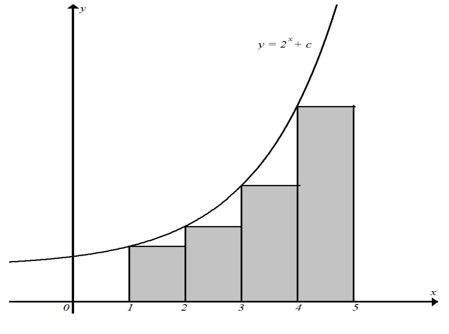
$Area \approx (5.4+5+2.4+0+1+0.6+0)\times1=14.4$ square units.

Example 7 The area under the curve $y=\sin(x)$ between $x=0$ and $x=\dfrac{\pi}{2}$ is approximated by two rectangles as shown. Find this approximation to the area.



$$Area \approx \left(\frac{\pi}{3}-\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)+\left(\frac{\pi}{2}-\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)=\frac{(1+\sqrt{3})\pi}{12}$$

Example 8 (2013 VCAA Exam 2)




Consider the graph of $y=2^x+c$, where $c$ is a real number. The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the $x$-axis and the lines $x=1$ and $x=5$. Find $c$ if the total area of the shaded rectangles is 44.

$(2^1+c)\times1+(2^2+c)\times1+(2^3+c)\times1+(2^4+c)\times1=44$
$(2+c)+(4+c)+(8+c)+(16+c)=44$
$30+4c=44$, $\therefore 4c=14$, $c=\dfrac{7}{2}$

**The fundamental theorem of calculus**

*If $F(x)$ is any anti-derivative of $f(x)$ on interval $[a,b]$, i.e. $F(x)=\int f(x)dx$, then $\int_a^b f(x)dx=F(b)-F(a)$.*

*This statement is known as the fundamental theorem of calculus.*

*$\int_a^b f(x)dx$ is called a **definite integral**, and $\int f(x)dx$, an anti-derivative of $f(x)$ is also called an **indefinite** integral.*

Example 9 Given the anti-derivative of $f(x)$ is $\sin x+\dfrac{x^2+1}{5}$, evaluate $\int_0^{\frac{\pi}{2}} f(x)dx$.

$$\int_0^{\frac{\pi}{2}} f(x)dx=\left(\sin\frac{\pi}{2}+\frac{\pi^2+4}{20}\right)-\left(\sin(0)+\frac{1}{5}\right)$$
$$=1+\frac{\pi^2+4}{20}-\frac{1}{5}=\frac{\pi^2}{20}+1$$

Example 10 Given the anti-derivative of $g(x)$ is $\dfrac{\log_e(x+1)}{x+1}$, evaluate $\int_0^1 g(x)dx$.

$$\int_0^1 g(x)dx=\frac{\log_e 2}{2}-\frac{\log_e 1}{1}=\frac{1}{2}\log_e 2 \text{ or } \log_e\sqrt{2}.$$

*An intermediate step is usually added before the substitutions of a and b.* $\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$.

Example 11 The anti-derivative of $h(x)$ is $\sin^2 x-\dfrac{1}{\cos^2 x}$, find an expression for $\int_0^t h(x)dx$ in terms of $t$, where $0<t<\dfrac{\pi}{2}$.

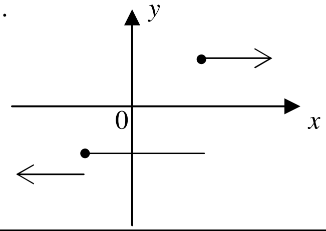
$$\int_0^t h(x)dx=\left[\sin^2 x-\frac{1}{\cos^2 x}\right]_0^t=\left(\sin^2 t-\frac{1}{\cos^2 t}\right)-\left(0-\frac{1}{\cos^2 0}\right)$$
$$=\sin^2 t-\frac{1}{\cos^2 t}+1.$$

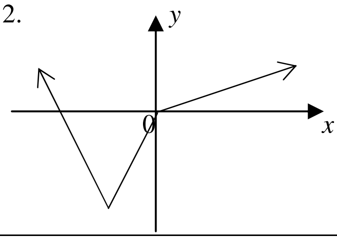
Example 12 Given the derivative of $\sin^{-1}(x^2)$ is $\dfrac{2x}{\sqrt{1-x^4}}$, evaluate $\int_0^{\frac{1}{\sqrt{2}}}\dfrac{2x}{\sqrt{1-x^4}}dx$.

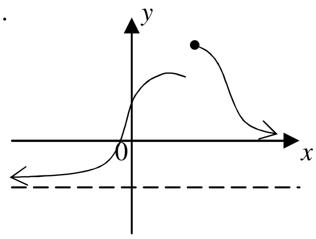
$$\int_0^{\frac{1}{\sqrt{2}}}\frac{2x}{\sqrt{1-x^4}}dx=\left[\sin^{-1}(x^2)\right]_0^{\frac{1}{\sqrt{2}}}=\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0)=\frac{\pi}{6}$$

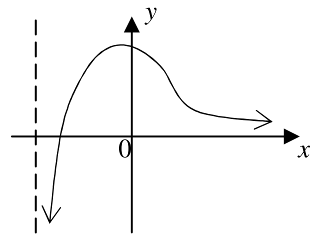
Exercise: Next page

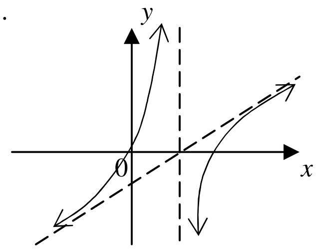
Q1 to 6. Deduce the graph of gradient function $f'(x)$ from the given graph of function $f(x)$.

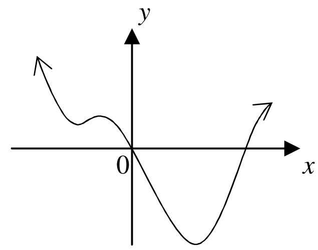
1.



2.


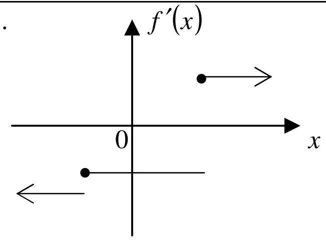

3.


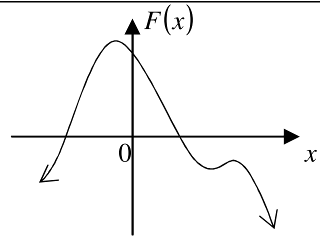

4.


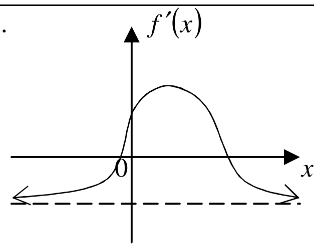

5.


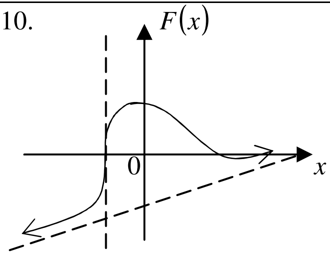

6.


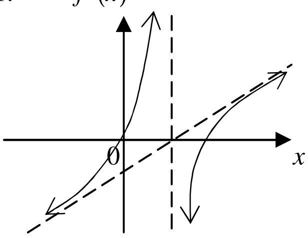

Q7 to 12. Deduce the graph of the original function $f(x)$ from the given graph of $f'(x)$ or an anti-derivative function $F(x)$.

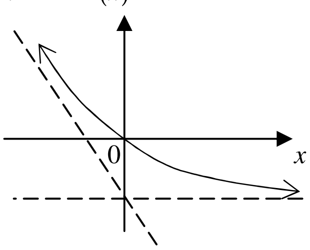
7.



8.



9.



10.



11. $f'(x)$



12. $F(x)$



13. $F(x) = \sin 3x - \cos 2x$ is an anti-derivative of $f(x)$ on $\left[0, \frac{\pi}{6}\right]$. Find the exact value of $\int_0^{\frac{\pi}{6}} f(x)dx$.

14. $F(x) = x\log_e|x|$ is an anti-derivative of $f(x)$ on $[-2,-1]$. Find the exact value of $\int_{-2}^{-1} f(x)dx$.

15. $F(x) = x^2 e^{-x}$ is an anti-derivative of $f(x)$ on $[-2,-1]$. Find the exact value of $\int_{-1}^{-2} f(x)dx$.

16. Use left rectangles of unit width to find the approximate area bounded by the curve $y = x^2 - 2x + 1$, the $x$-axis, the lines $x = -1$ and $x = 3$.

Answers: Q 13 1.5 Q14 $2\log_e 2$ Q15 $4e^2 - e$ Q16 6