

Properties of anti-derivatives and definite integrals

$$\int A f(x) dx = A \int f(x) dx$$

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\int_a^a f(x) dx = 0$$

Rules for anti-derivatives of x^n , where $n \in Q$

For $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$$

$$\int (x-b)^n dx = \frac{(x-b)^{n+1}}{n+1} + C$$

$$\int a(kx-b)^n dx = \frac{a(kx-b)^{n+1}}{k(n+1)} + C$$

For $n = -1$, i.e. x^{-1} or $\frac{1}{x}$, where $x \neq 0$,

$$\int \frac{1}{x} dx = \log_e |x| + C$$

$$\int \frac{a}{kx} dx = \frac{a}{k} \log_e |x| + C$$

$$\int \frac{1}{x-b} dx = \log_e |x-b| + C$$

$$\int \frac{a}{k(x-b)} dx = \frac{a}{k} \log_e |x-b| + C$$

$$\int \frac{a}{kx-b} dx = \frac{a}{k} \log_e |kx-b| + C$$

Rules for anti-derivative of e^{kx}

$$\int e^x dx = e^x + C$$

$$\int ae^{kx} dx = \frac{a}{k} e^{kx} + C$$

$$\int e^{x-b} dx = e^{x-b} + C$$

$$\int ae^{k(x-b)} dx = \frac{a}{k} e^{k(x-b)} + C$$

$$\int ae^{kx-b} dx = \frac{a}{k} e^{kx-b} + C$$

Rules for anti-derivatives of $\cos(kx)$ and $\sin(kx)$

$$\int \cos(x) dx = \sin(x) + C$$

$$\int a \cos(kx) dx = \frac{a}{k} \sin(kx) + C$$

$$\int \cos(x-b) dx = \sin(x-b) + C$$

$$\int a \cos k(x-b) dx = \frac{a}{k} \sin k(x-b) + C$$

$$\int a \cos(kx-b) dx = \frac{a}{k} \sin(kx-b) + C$$

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int a \sin(kx) dx = -\frac{a}{k} \cos(kx) + C$$

$$\int \sin(x-b) dx = -\cos(x-b) + C$$

$$\int a \sin(k(x-b)) dx = -\frac{a}{k} \cos k(x-b) + C$$

$$\int a \sin(kx-b) dx = -\frac{a}{k} \cos(kx-b) + C$$

Example 1 Find *an* anti-derivative of $\frac{-3}{5(2x+1)^2}$.

$$\int \frac{-3}{5(2x+1)^2} dx = \int \frac{-3(2x+1)^{-2}}{5} dx = \frac{-3(2x+1)^{-1}}{5(-1)(2)} + C = \frac{3}{10(2x+1)} + C$$

Choose any real value for C , usually 0 for convenience.

Example 2 Find $F(x) = \int \frac{2}{3x-4} dx$ such that $F(1) = 1$.

$$F(x) = \frac{2}{3} \log_e |3x-4| + C, F(1) = \frac{2}{3} \log_e |-1| + C = 1,$$

$$\therefore \frac{2}{3} \log_e (1) + C = 1, C = 1. \therefore F(x) = \frac{2}{3} \log_e |3x-4| + 1$$

Example 3 Given $f'(x) = \cos \pi(x+1) - 2e^{\pi x}$, find $f(x)$ such that

$$f(0) = -\frac{2}{\pi}.$$

$$f'(x) = \cos \pi(x+1) - 2e^{\pi x}, \therefore f(x) = \int (\cos \pi(x+1) - 2e^{\pi x}) dx$$

$$= \frac{1}{\pi} \sin \pi(x+1) - \frac{2}{\pi} e^{\pi x} + C.$$

$$f(0) = \frac{1}{\pi} \sin \pi - \frac{2}{\pi} e^0 + C = -\frac{2}{\pi}, \therefore C = 0.$$

$$\therefore f(x) = \frac{1}{\pi} \sin \pi(x+1) - \frac{2}{\pi} e^{\pi x}$$

Example 4 Evaluate $\int_{-1}^1 (x^2 - 1)(x^2 + 1) dx$.

$$\begin{aligned} \int_{-1}^1 (x^2 - 1)(x^2 + 1) dx &= \int_{-1}^1 (x^4 - 1) dx = \left[\frac{x^5}{5} - x \right]_{-1}^1 \\ &= \left(\frac{1}{5} - 1 \right) - \left(-\frac{1}{5} + 1 \right) = -\frac{8}{5} \end{aligned}$$

Example 5 Evaluate $\int_0^{\log_e 2} \frac{e^{2x} - e^{-2x}}{e^{2x}} dx$.

$$\begin{aligned} \int_0^{\log_e 2} \frac{e^{2x} - e^{-2x}}{e^{2x}} dx &= \int_0^{\log_e 2} (1 - e^{-4x}) dx = \left[x + \frac{1}{4} e^{-4x} \right]_0^{\log_e 2} \\ &= \left(\log_e 2 + \frac{1}{4} e^{-4 \log_e 2} \right) - \left(0 + \frac{1}{4} e^0 \right) = \log_e 2 + \frac{1}{4} \times \frac{1}{16} - \frac{1}{4} \\ &= \log_e 2 - \frac{15}{64} \end{aligned}$$

Example 6 Evaluate $\int_0^1 \frac{x^2+1}{x+1} dx$.

$$\begin{aligned}\int_0^1 \frac{x^2+1}{x+1} dx &= \int_0^1 \left(x-1 + \frac{2}{x+1} \right) dx = \left[\frac{x^2}{2} - x + 2 \log_e(x+1) \right]_0^1 \\ &= \left(\frac{1}{2} - 1 + 2 \log_e 2 \right) - (0) = 2 \log_e 2 - \frac{1}{2}\end{aligned}$$

Example 7 Evaluate $\int_1^3 \left(\frac{1+x^3}{x+1} \right) dx$.

$$\begin{aligned}\int_1^3 \left(\frac{1+x^3}{x+1} \right) dx &= \int_1^3 \frac{(x+1)(x^2-x+1)}{x+1} dx = \int_1^3 (x^2-x+1) dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} + x \right]_1^3 = \frac{20}{3}\end{aligned}$$

Example 8 Evaluate $\int_{11}^{12} \frac{1}{10^{2\log_{10}(x-10)}} dx$.

Firstly simplify the integrand:

$$\begin{aligned}\frac{1}{10^{2\log_{10}(x-10)}} &= \frac{1}{(x-10)^2} = (x-10)^{-2} \\ \int_{11}^{12} \frac{1}{10^{2\log_{10}(x-10)}} dx &= \int_{11}^{12} (x-10)^{-2} dx = \left[-\frac{1}{x-10} \right]_{11}^{12} = \frac{1}{2}\end{aligned}$$

Integration by recognition

Example 9 Find $\int \frac{1}{\cos^2 x} dx$.

$$\begin{aligned}\int \frac{1}{\cos^2(x)} dx &= \int \sec^2(x) dx = \tan(x) + C \text{ by recognising that} \\ \frac{d}{dx}(\tan(x)) &= \sec^2(x)\end{aligned}$$

Example 10 Find the derivative of $x \log_e x$. Hence find the anti-derivative of $\log_e x$.

Let $y = x \log_e x$, apply the product rule to obtain

$$\begin{aligned}\frac{dy}{dx} &= (\log_e x)(1) + (x)\left(\frac{1}{x}\right) = \log_e x + 1, \\ \therefore \log_e x &= \frac{dy}{dx} - 1. \int \log_e x dx = \int \left(\frac{dy}{dx} - 1 \right) dx = \int \frac{dy}{dx} dx - \int 1 dx \\ &= y - x + C = x \log_e x - x + C\end{aligned}$$

Example 11 Find $f'(x)$, given $f(x) = \sin(x^{-1})$.

Hence evaluate $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(x^{-1})}{x^2} dx$.

$$\begin{aligned}f(x) &= \sin(x^{-1}), \text{ use the chain rule to obtain } f'(x) = -x^{-2} \cos(x^{-1}) \\ \therefore \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(x^{-1})}{x^2} dx &= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} -f'(x) dx = \left[-f(x) \right]_{\frac{\pi}{2}}^{\frac{\pi}{2}} = \left[-\sin(x^{-1}) \right]_{\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \left(-\sin\left(\frac{\pi}{2}\right) \right) - \left(-\sin\left(-\frac{\pi}{2}\right) \right) = -1 - 1 = -2\end{aligned}$$

Example 12 Given $y = xe^{-2x}$, find $\frac{dy}{dx}$ and hence evaluate

$$\int_0^1 xe^{-2x} dx.$$

$$\begin{aligned}y &= xe^{-2x}, \therefore \frac{dy}{dx} = (x)(-2e^{-2x}) + (1)(e^{-2x}) = -2xe^{-2x} + e^{-2x} \\ \therefore xe^{-2x} &= \frac{1}{2} \left(e^{-2x} - \frac{dy}{dx} \right), \int_0^1 xe^{-2x} dx = \int_0^1 \frac{1}{2} \left(e^{-2x} - \frac{dy}{dx} \right) dx \\ &= \left[\frac{1}{2} \left(\frac{e^{-2x}}{-2} - xe^{-2x} \right) \right]_0^1 = \frac{1}{2} \left(\frac{e^{-2}}{-2} - e^{-2} \right) - \frac{1}{2} \left(\frac{1}{-2} - 0 \right) \\ &= \frac{1}{4} - \frac{3e^{-2}}{4} = \frac{1}{4} (1 - 3e^{-2})\end{aligned}$$

Example 13 Let $f(x) = x^2 + 1$ and $g(x) = \log_e(x)$.

a. Write down the rule of $g(f(x))$.

b. Find the derivative of $g(f(x))$.

c. Hence find an anti-derivative of $\frac{x}{x^2+1}$.

$$a. g(f(x)) = \log_e(f(x)) = \log_e(x^2+1)$$

$$b. \frac{d}{dx} g(f(x)) = \frac{d}{dx} \log_e(x^2+1) = \frac{1}{x^2+1} \times \frac{d}{dx}(x^2+1) = \frac{2x}{x^2+1}$$

$$c. \int \frac{2x}{x^2+1} dx = \log_e(x^2+1) + C, \therefore \int \frac{x}{x^2+1} dx = \frac{1}{2} \log_e(x^2+1)$$

Example 14

If $f(x) = x \cos(3x)$, then $f'(x) = \cos(3x) - 3x \sin(3x)$.

Use this fact to find an anti-derivative of $x \sin(3x)$.

$$f'(x) = \cos(3x) - 3x \sin(3x),$$

$$\therefore \int f'(x) dx = \int (\cos(3x) - 3x \sin(3x)) dx$$

$$\int f'(x) dx = \int \cos(3x) dx - 3 \int x \sin(3x) dx$$

$$f(x) = \frac{\sin(3x)}{3} - 3 \int x \sin(3x) dx$$

$$3 \int x \sin(3x) dx = \frac{\sin(3x)}{3} - x \cos(3x)$$

$$\int x \sin(3x) dx = \frac{\sin(3x)}{9} - \frac{x \cos(3x)}{3}$$

Example 15

a. Show that $\frac{d}{dx}(x+n)e^x = (x+n+1)e^x$, where n is an integer. b.

Hence find $\int (x+n)e^x dx$.

a. By the product rule,

$$\begin{aligned}\frac{d}{dx}(x+n)e^x &= \left(\frac{d}{dx}(x+n) \right) e^x + (x+n) \frac{d}{dx} e^x \\ &= e^x + (x+n)e^x = (x+n+1)e^x\end{aligned}$$

$$(x+n)e^x = \int (x+n+1)e^x dx = \int (x+n)e^x dx + \int e^x dx$$

$$\therefore (x+n)e^x = \int (x+n+1)e^x dx = \int (x+n)e^x dx + e^x$$

$$\therefore \int (x+n)e^x dx = (x+n)e^x - e^x = (x+n-1)e^x$$

1. Evaluate $\int_{-3}^{-1} \frac{2}{1-x} dx$.

3. Given $f(x) = 2\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)$, find $\int f(x)dx$.

5. Evaluate $\int_0^{\frac{1}{2}} \left[\frac{d}{dx} \left(\frac{1}{x^2 - x + 1} \right) \right] dx$.

7. Given $\frac{d}{dx} \left(\frac{\log_e x}{x} \right) = \frac{1 - \log_e x}{x^2}$, evaluate $\int_1^e \left(\frac{1 - \log_e x}{x^2} \right) dx$.

9. Find the derivative of $\sqrt{\cos(2x)}$. Hence evaluate

$$\int_0^{\frac{\pi}{6}} \sqrt{\sin(2x)\tan(2x)} dx.$$

11. Find $\frac{d}{dx} \left(\sqrt{1+x^2} \right)$ and hence find $\int_0^t \frac{x}{\sqrt{1+x^2}} dx$.

2. Find $\int (\sqrt{3x-2})^{-1} dx$.

4. Evaluate $\int_0^{-\pi} \sin\left(\frac{x}{2}\right) dx + \int_0^{\pi} \sin\left(\frac{x}{2}\right) dx$.

6. Given $\int_0^{\pi/2} g(x) dx = \pi$, evaluate $\int_0^{\pi/2} [\cos\left(\frac{x}{2\pi}\right) - 2g(x)] dx$.

8. Show that $\frac{d}{dx} (\sec(2x) + 1) = 2\sec(2x)\tan(2x)$. Hence find $\int (\sec(2x)\tan(2x)) dx$.

10. Find $f(x)$, given $f'(x) = \sec^2(2x)$ and $f\left(\frac{\pi}{8}\right) = 1$.

12. Find $f'(x)$, given $f(x) = x \sin x$. Hence find $\int x \cos x dx$.