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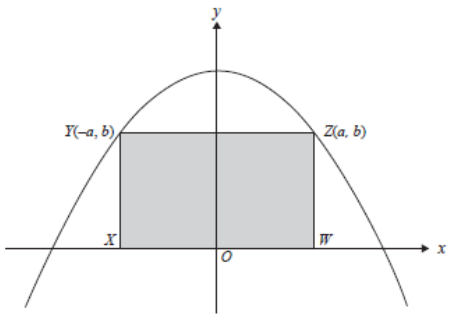
Maximum/minimum problems

In practical situations, the value of a quantity (dependent variable) varies with the value of another quantity (independent variable). For certain value of the latter, the value of the former is a maximum/minimum. This can be found by differentiation or graphics calculator.

In formulating a problem, sometimes the dependent variable is a function of two other variables. By establishing the relationship between these variables the dependent variable can then be expressed in terms of one of them before the process of finding the maximum/minimum.

Example 1 (2006 VCAA Exam 1)

A rectangle $XYZW$ has two vertices, X and W, on the x-axis and the other two vertices, Y and Z, on the graph of $y = 9 - 3x^2$, as shown in the diagram below. The coordinates of Z are (a, b) where a and b are positive real numbers.



a. Find the area, A, of rectangle $XYZW$ in terms of a.

$b = 9 - 3a^2$, $A = 2ab = 2a(9 - 3a^2) = 18a - 6a^3$, $a > 0$

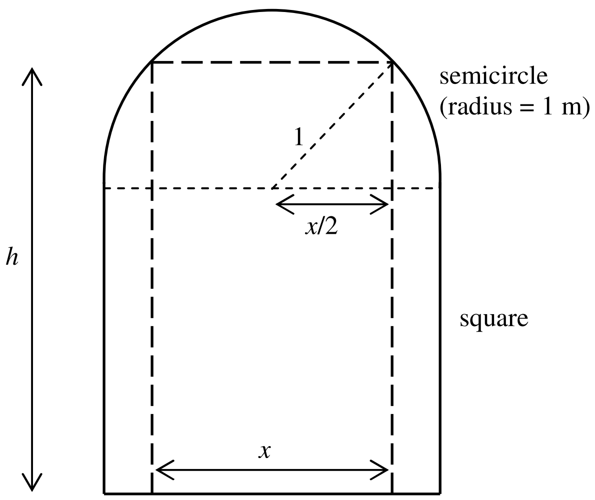
b. Find the maximum value of A and the value of a for which this occurs.

$\frac{dA}{da} = 18 - 18a^2$

Let $\frac{dA}{da} = 0$ to find max A. $18 - 18a^2 = 0$, $a = 1$

.: $A_{\max} = 12$

Example 2 An upright rectangular window is to be installed in the middle of a wall with dimensions shown in the following figure.



Find the area of the largest window possible for the wall.
Let x m and h m be the width and the height of the window respectively.
Area of the window $A = xh$

The relationship between h and x: $h = 2 + \sqrt{1 - \left(\frac{x}{2}\right)^2}$

.: $A(x) = x\left(2 + \sqrt{1 - \left(\frac{x}{2}\right)^2}\right) = x\left(2 + \sqrt{1 - \frac{x^2}{4}}\right)$

Use graphics calculator to graph the area function $A(x)$ and determine maximum $A = 4.40$ m^2 when $x = 1.86$ m.

Example 3 Find the points on the curve $y = x^2 + 2x - 1$ closest to the point $(-1,-1)$.

Let (x, y) be a point on the curve closest to $(-1,-1)$. The distance between the two points is:

$D(x) = \sqrt{(x - {}^{-}1)^2 + (y - {}^{-}1)^2} = \sqrt{(x+1)^2 + (x^2 + 2x)^2}$.

$\frac{dD}{dx} = \frac{1}{2\sqrt{(x+1)^2 + (x^2 + 2x)^2}}\left(2(x+1) + 2(x^2 + 2x)(2x + 2)\right)$.

Let $\frac{dD}{dx} = 0$, .: $2(x+1) + 2(x^2 + 2x)(2x + 2) = 0$,

$2(x+1)(1 + 2(x^2 + 2x)) = 0$, $x = -1$ and $y = -2$ or

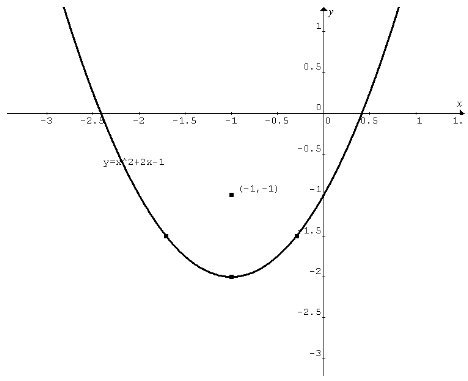
$2x^2 + 4x + 1 = 0$, i.e. $x = \frac{-4 + \sqrt{16 - 8}}{4} = \frac{-2 + \sqrt{2}}{2}$ and

$y = \left(\frac{-2 + \sqrt{2}}{2}\right)^2 + 2\left(\frac{-2 + \sqrt{2}}{2}\right) - 1 = \frac{3 - 2\sqrt{2}}{2} - 2 + \sqrt{2} - 1 = -\frac{3}{2}$

or $x = \frac{-4 - \sqrt{16 - 8}}{4} = \frac{-2 - \sqrt{2}}{2}$ and

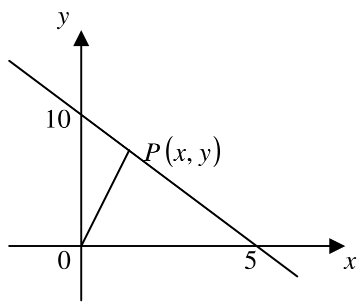
$y = \left(\frac{-2 - \sqrt{2}}{2}\right)^2 + 2\left(\frac{-2 - \sqrt{2}}{2}\right) - 1 = \frac{3 + 2\sqrt{2}}{2} - 2 - \sqrt{2} - 1 = -\frac{3}{2}$.

The three points closest to $(-1,-1)$ are $(-1,-2)$, $\left(\frac{-2 + \sqrt{2}}{2}, -\frac{3}{2}\right)$ and $\left(\frac{-2 - \sqrt{2}}{2}, -\frac{3}{2}\right)$.



Example 4 (2007 VCAA Exam 1)

P is the point on the line $2x + y - 10 = 0$ such that the length of OP, the line segment from the origin O to P, is a minimum. Find the coordinates of P and this minimum length.



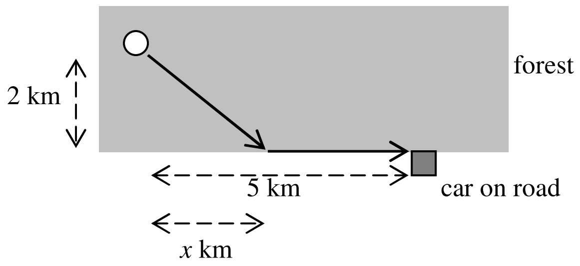
$y = -2x + 10$, $L = \sqrt{x^2 + y^2}$

$\therefore\ L=\sqrt{x^2+(-2x+10)^2}=\sqrt{5x^2-40x+100}$

$\frac{dL}{dx}=\frac{5x-20}{\sqrt{5x^2-40x+100}}$

Let $\frac{dL}{dx}=0$ to find min L. $5x-20=0$, $x=4$, $y=2$, $L_{\min}=2\sqrt{5}$

Example 5 A bushwalker is in a forest 2 km from a straight road. His car is 5 km down the road. He can walk 6 km h^{-1} in the forest and 8 km h^{-1} along the road. Toward what point on the road should he walk to minimise the time required to reach his car?



Let x km be the distance of the bushwalker from the point along the road. He walks $\sqrt{x^2+2^2}$ km in the forest and $5-x$ km along the road. Using the formula $\text{time}=\frac{\text{distance}}{\text{speed}}$, the total time taken

$T(x)=\frac{\sqrt{x^2+4}}{6}+\frac{5-x}{8}$ hours, where $0\le x\le 5$.

$T'(x)=\frac{1}{6\times 2\sqrt{x^2+4}}\times 2x-\frac{1}{8}=\frac{x}{6\sqrt{x^2+4}}-\frac{1}{8}$.

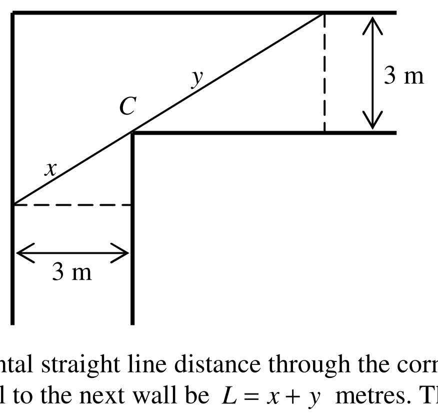
Let $T'(x)=0$, $\frac{x}{6\sqrt{x^2+4}}-\frac{1}{8}=0$, $6\sqrt{x^2+4}=8x$,

$3\sqrt{x^2+4}=4x$, $9(x^2+4)=16x^2$, $7x^2=36$, $\therefore\ x=\frac{6}{\sqrt{7}}=\frac{6\sqrt{7}}{7}$

$x=$	2	$\frac{6\sqrt{7}}{7}$	3
$\frac{dT}{dx}$	<0	$=0$	>0
Nature		Local min.	

$\therefore$ the point that minimise the time is $\frac{6\sqrt{7}}{7}$ km along the road from the initial position of the bushwalker.

Example 6 Find the length of the longest rigid pipe that can be carried horizontally from a 3-m wide corridor to another 3-m wide corridor perpendicular to the first. Ignore the diameter of the pipe.



Let the horizontal straight line distance through the corner C from one inside wall to the next wall be $L=x+y$ metres. There is a relationship between x and y. Similar triangles, $\frac{y}{x}=\frac{3}{\sqrt{x^2-9}}$,

$\therefore\ y=\frac{3x}{\sqrt{x^2-9}}$, $\therefore\ L(x)=x+\frac{3x}{\sqrt{x^2-9}}$.

The longest possible pipe is restricted by the shortest distance L.

$\frac{dL}{dx}=1+\frac{3\sqrt{x^2-9}-(3x)\left(\frac{x}{\sqrt{x^2-9}}\right)}{x^2-9}$

$=1+\frac{3(x^2-9)-3x^2}{(x^2-9)\sqrt{x^2-9}}=1-\frac{27}{(x^2-9)^{\frac{3}{2}}}$. Let $\frac{dL}{dx}=0$,

$1-\frac{27}{(x^2-9)^{\frac{3}{2}}}=0$, $\frac{27}{(x^2-9)^{\frac{3}{2}}}=1$, $(x^2-9)^{\frac{1}{2}}=3$, $x^2-9=9$,

$\therefore\ x=3\sqrt{2}$ and $y=\frac{9\sqrt{2}}{3}=3\sqrt{2}$. Hence $L=6\sqrt{2}$.

$x=$	4	$3\sqrt{2}$	5
$\frac{dL}{dx}$	<0	$=0$	>0
Nature		Local min.	

$\therefore\ L=6\sqrt{2}$ is the shortest distance and hence the longest pipe is $6\sqrt{2}$ m.

Example 7 The concentration y of a drug in the blood time t after injection is given by $y=\frac{k}{a-b}\left(e^{-bt}-e^{-at}\right)$, where a, b and k are positive constants and $a>b$.
Find the maximum concentration and the time it occurs after injection.

$\frac{dy}{dt}=\frac{k}{a-b}\left(-be^{-bt}+ae^{-at}\right)$. Let $\frac{dy}{dt}=0$,

$\frac{k}{a-b}\left(-be^{-bt}+ae^{-at}\right)=0$, $\therefore\ -be^{-bt}+ae^{-at}=0$, $\frac{e^{-bt}}{e^{-at}}=\frac{a}{b}$,

$e^{at-bt}=\frac{a}{b}$, $e^{(a-b)t}=\frac{a}{b}$, $(a-b)t=\log_e\left(\frac{a}{b}\right)$, $t=\frac{1}{a-b}\log_e\left(\frac{a}{b}\right)$.

At $t<\frac{1}{a-b}\log_e\left(\frac{a}{b}\right)$, $(a-b)t<\log_e\left(\frac{a}{b}\right)$, $e^{(a-b)t}<\frac{a}{b}$,

$\frac{e^{-bt}}{e^{-at}}<\frac{a}{b}$, $\therefore\ be^{-bt}-ae^{-at}<0$, $\therefore\ -be^{-bt}+ae^{-at}>0$, $\therefore\ \frac{dy}{dt}>0$.

Similarly, at $t>\frac{1}{a-b}\log_e\left(\frac{a}{b}\right)$, $\frac{dy}{dt}<0$.

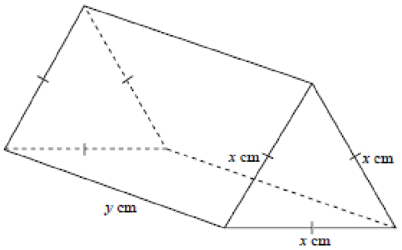
$\therefore$ at $t=\frac{1}{a-b}\log_e\left(\frac{a}{b}\right)$, y is maximum.

$y_{\max}=\frac{k}{a-b}\left(e^{-\frac{b}{a-b}\ln\left(\frac{a}{b}\right)}-e^{-\frac{a}{a-b}\ln\left(\frac{a}{b}\right)}\right)$

Exercise: Next page

Q1 (2008 VCAA Exam 1)

A plastic brick is made in the shape of a right triangular prism. The triangular end is an equilateral triangle with side length x cm and the length of the brick is y cm.



The volume of the brick is 1000 cm^3.

a. Find an expression for y in terms of x.

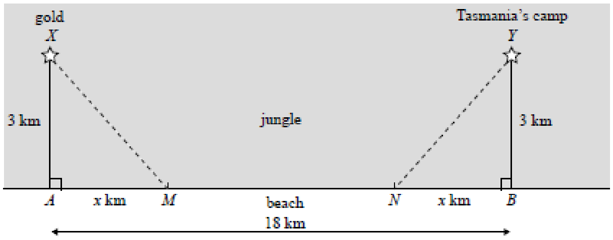
b. Show that the total surface area, A cm^2, of the brick is given by

$$A = \frac{4000\sqrt{3}}{x} + \frac{\sqrt{3}x^2}{2}$$

c. Find the value of x for which the brick has minimum total surface area. (You do not have to find this minimum.)

Q2 (2008 VCAA Exam 2)

Tasmania Jones is in the jungle, digging for gold. He finds the gold at X which is 3 km from a point A. Point A is on a straight beach.
Tasmania's camp is at Y which is 3 km from a point B. Point B is also on the straight beach.
$AB = 18$ km and $AM = NB = x$ km and $AX = BY = 3$ km.



While he is digging up the gold, Tasmania is bitten by a snake which injects toxin into his blood. After he is bitten, the concentration of the toxin in his bloodstream increases over time according to the equation

$$y = 50\log_e(1 + 2t)$$

where y is the concentration, and t is the time in hours after the snake bites him.

The toxin will kill him if its concentration reaches 100.

a. Find the time, to the nearest minute, that Tasmania has to find an antidote (that is, a cure for the toxin).

Tasmania has an antidote to the toxin at his camp. He can run through the jungle at 5 km/h and he can run along the beach at 13 km/h.

b. Show that he will not get the antidote in time if he runs directly to his camp through the jungle.

In order to get the antidote, Tasmania runs through the jungle to M on the beach, runs along the beach to N and then runs through the jungle to the camp at Y. M is x km from A and N is x km from B. (See diagram.)

c. Show that the time taken to reach the camp, T hours, is given by

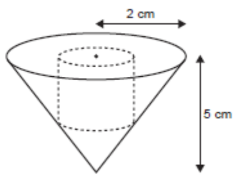
$$T = 2\left(\frac{\sqrt{9+x^2}}{5} + \frac{9-x}{13}\right)$$

d. Find the value of x which allows Tasmania to get to his camp in the minimum time.

e. Show that he gets to his camp in time to get the antidote.

Q3 (2010 VCAA Exam 1)

A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane as shown in the diagram below. The height of the cone is 5 cm and the radius of the cone is 2 cm. The radius of the cylinder is r cm and the height of the cylinder is h cm.



For the cylinder inscribed in the cone as shown above

a. find h in terms of r

The total surface area, S cm^2, of a cylinder of height h cm and radius r cm is given by the formula

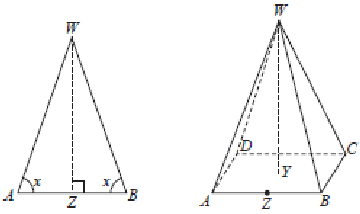
$$S = 2\pi rh + 2\pi r^2.$$

b. find S in terms of r

c. find the value of r for which S is a maximum.

Q4 (2010 VCAA Exam 2)

An ancient civilisation buried its kings and queens in tombs in the shape of a square-based pyramid, $WABCD$.
The kings and queens were each buried in a pyramid with $WA = WB = WC = WD = 10$ m.
Each of the isosceles triangle faces is congruent to each of the other triangular faces.
The base angle of each of these triangles is x, where $\frac{\pi}{4} < x < \frac{\pi}{2}$.
Pyramid $WABCD$ and a face of the pyramid, WAB, are shown here.



Z is the midpoint of AB.

a. i. Find AB in terms of x.

ii. Find WZ in terms of x.

b. Show that the total surface area (including the base), S m^2, of the pyramid, $WABCD$, is given by $S = 400(\cos^2(x) + \cos(x)\sin(x))$.

c. Find WY, the height of the pyramid $WABCD$, in terms of x.

d. The volume of any pyramid is given by the formula Volume $= \frac{1}{3} \times$ area of base $\times$ vertical height.

Show that the volume, T m^3, of the pyramid $WABCD$ is $\frac{4000}{3}\sqrt{\cos^4 x - 2\cos^6 x}$.

Queen Hepzabah's pyramid was designed so that it had the **maximum possible volume**.

e. Find $\frac{dT}{dx}$ and hence find the exact volume of Queen Hepzabah's pyramid and the corresponding value of x.

Queen Hepzabah's daughter, Queen Jepzibah, was also buried in a pyramid. It also had

$$WA = WB = WC = WD = 10 \text{ m}.$$

The volume of Jepzibah's pyramid is exactly one half of the volume of Queen Hepzabah's pyramid. The volume of Queen Jepzibah's pyramid is also given by the formula for T obtained in **part d.**

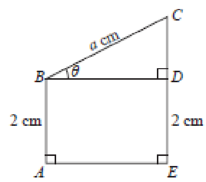
f. Find the possible values of x, for Jepzibah's pyramid, correct to two decimal places.

Q5 (2011 VCAA Exam 1)

The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. The point D is on line segment EC with $AB = ED = 2$ cm and $BC = a$ cm, where a is a positive constant.

$$\angle BAE = \angle CEA = \frac{\pi}{2}$$

Let $\angle CBD = \theta$ where $0 < \theta < \frac{\pi}{2}$.



a. Find BD and CD in terms of a and θ.

b. Find the length, L cm, of the wire in the frame, including length BD, in terms of a and θ.

c. Find $\frac{dL}{d\theta}$, and hence show that $\frac{dL}{d\theta} = 0$ when $BD = 2CD$.

d. Find the maximum value of L if $a = 3\sqrt{5}$.

Answers: 1a $y = \frac{4000\sqrt{3}}{3x^2}$ 1c $x = \sqrt[3]{4000}$

2a 3 hours 12 minutes 2d $x = \frac{5}{4}$

3a $h = 5\left(1 - \frac{r}{2}\right)$ 3b $S = 10\pi r - 3\pi r^2$ 3c $r = \frac{5}{3}$

4ai $20\cos(x)$ 4aii $10\sin(x)$ 4c $10\sqrt{1 - 2\cos^2(x)}$

4e $x = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $T_{\max} = \frac{4000\sqrt{3}}{27}$ m^3

4f $x \approx 0.81$ or 1.23 radians

5a $BD = a\cos\theta$, $CD = a\sin\theta$ 5b $4 + a + a\sin\theta + 2a\cos\theta$

5c $\frac{dL}{d\theta} = a\cos\theta - 2a\sin\theta$ 5d $L_{\max} = 19 + 3\sqrt{5}$