Rate of change of a function

If $f(x)$ is any function differentiable at $x = a$, then the rate of change of $f(x)$ with respect to x at $x = a$ is $f'(a)$.

The average rate of change of $f(x)$ with respect to x between $x = a$ and $x = b$, where $b > a$, is $\frac{f(b) - f(a)}{b - a}$.

Example 1 The position of a particle moving along a straight line at time *t* is $r(t) = t^2 - 2t - 3$, where $0 \le t \le 6$.

Given the initial velocity $v(0) = {}^-2$, find
(a) the initial position of the particle,
(b) the rate of change of position with respect to time (i.e. the velocity) at $t = 1$,
(c) the average rate of change of position with respect to time (i.e. average velocity) between $t = 0$ and $t = 5$,
(d) the rate of change of velocity with respect to time (i.e. the acceleration) at $t = 1$,
(e) the average rate of change of velocity with respect to time (i.e. the average acceleration) between $t = 0$ and $t = 5$,
(f) sketch the *r-t* graph,
(g) use the graph to find the distance travelled from $t = 0$ to $t = 5$,
(h) the average rate of change in distance travelled with respect to time (i.e. the average speed) between $t = 0$ and $t = 5$.

(a) Initial position: at $t = 0$, $r(0) = {}^-3$.

$$\xrightarrow[\quad {}^-3 \qquad\qquad 0 \qquad\qquad\qquad r]{}$$

(b) Velocity: $v(t) = \frac{dr}{dt} = 2t - 2$, at $t = 1$, $v(1) = 0$, i.e. the particle is momentarily at rest.

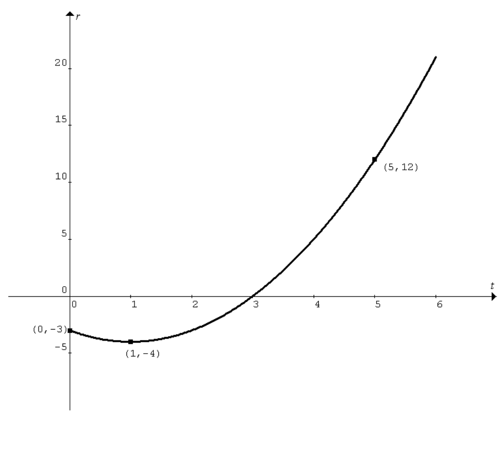
(c) Average velocity: $v_{av} = \frac{r(5) - r(0)}{5 - 0} = \frac{{}^+12 - {}^-3}{5} = {}^+3$

(d) Acceleration: $a = \frac{dv}{dt} = {}^+2$, which is a constant. Same acceleration for $t > 0$.

(e) Average acceleration: $a_{av} = \frac{v(5) - v(0)}{5 - 0} = \frac{{}^+8 - {}^-2}{5} = {}^+2$.

a and a_{av} are the same because *a* is constant.

(f)



(g) The particle moves backwards from ${}^-3$ to ${}^-4$, then forwards to ${}^+12$. Total distance $= 1 + 16 = 17$.

(h) Average speed $= \frac{\text{distance}}{\text{time}} = \frac{17}{5} = 3.4$, which is different from the average **velocity**.

Example 2
(a) Find the rate of change of volume *V* of a spherical balloon with respect to its radius *r* when $r = 5$.
(b) Find the rate of change of volume *V* of the spherical balloon with respect to its surface area *A* when $r = 5$.

(a) Volume of a sphere $V(r) = \frac{4}{3}\pi r^3$. $\frac{dV}{dr} = 4\pi r^2$, when $r = 5$, $\frac{dV}{dr} = 4\pi(5^2) = 100\pi.$

(b) Surface area $A = 4\pi r^2$, $\therefore r = \sqrt{\frac{A}{4\pi}}$,

$\therefore V(A) = \frac{4\pi}{3}\left(\sqrt{\frac{A}{4\pi}}\right)^3 = \frac{1}{3\sqrt{4\pi}} A^{\frac{3}{2}} = \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}}$.

$\frac{dV}{dA} = \frac{3}{2} \times \frac{1}{6\sqrt{\pi}} A^{\frac{1}{2}} = \frac{1}{4}\sqrt{\frac{A}{\pi}}$, when $r = 5$, $A = 100\pi$, $\therefore \frac{dV}{dA} = \frac{5}{2}$.

Note: (b) can be done in the following way since the two rates, $\frac{dV}{dr}$ and $\frac{dV}{dA}$ are related.

$V(r) = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2$. $A(r) = 4\pi r^2$, $\frac{dA}{dr} = 8\pi r$.

Using the chain rule: $\frac{dV}{dr} = \frac{dV}{dA} \times \frac{dA}{dr}$,

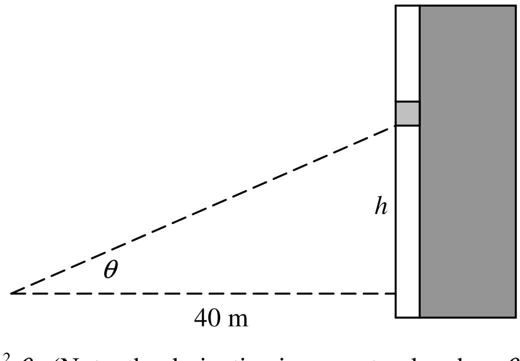
$\therefore \frac{dV}{dA} = \frac{dV}{dr} \Big/ \frac{dA}{dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$. When $r = 5$, $\frac{dV}{dA} = \frac{5}{2}$.

Related rates

If two quantities x and y are related by $y = f(x)$, then the rate of change of y with respect to a third quantity t is related to the rate of change of x with respect to the third quantity t by $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ or $\frac{dy}{dt} = f'(x) \times \frac{dx}{dt}$.

Example 3 At a distance of 40 m from the base of a tall building, a person observes an external elevator moving up at a constant speed of 2.0 m s^{-1}. Determine the rate of change of the angle of elevation when the elevator is 30 m above the person's eye level.

The relationship between the height above the eye level and the angle of elevation is $h = 40\tan\theta$.

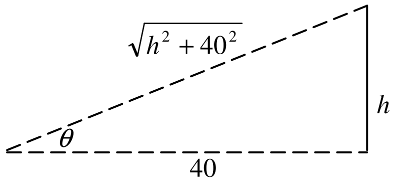


$\frac{dh}{d\theta} = 40\sec^2\theta$. (Note: the derivative is correct only when θ is in radians)

The chain rule: $\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt}$,

$\therefore \frac{d\theta}{dt} = \frac{dh}{dt} \Big/ \frac{dh}{d\theta} = \frac{2.0}{40\sec^2\theta} = \frac{1}{20}\cos^2\theta$.

Instead of finding θ when $h = 30$, and substitute into the above expression, find $\cos\theta$ in terms of h.

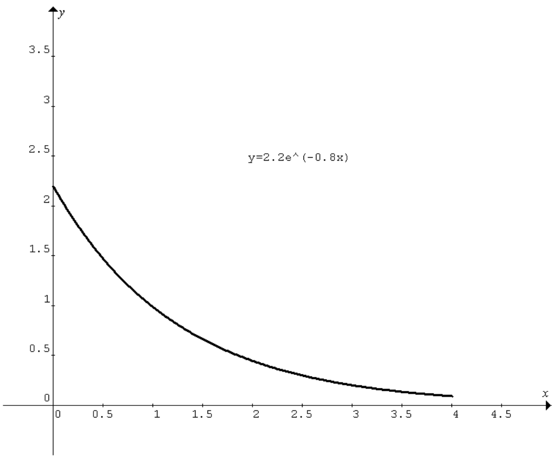


$\therefore\ \cos\theta = \dfrac{40}{\sqrt{h^2+40^2}}$ and $\dfrac{d\theta}{dt} = \dfrac{1}{20}\left(\dfrac{40}{\sqrt{h^2+40^2}}\right)^2$

When $h = 30$, $\dfrac{d\theta}{dt} = 0.032$ radians per second

$= 0.032 \times \dfrac{180^o}{\pi} \approx 1.8$ degrees per second.

Example 4 The side elevation of a playground slide can be modelled by $y = 2.2e^{-0.8x}$ for $0 \le x \le 4$. Length measures are in metres and time in seconds. The shadow on the ground of a girl sliding down moves at 1.5 m s^{-1} when she is at $x = 2.0$. Calculate her descending speed (vertical) at that moment.



$y = 2.2e^{-0.8x}$, $\dfrac{dy}{dx} = -1.76e^{-0.8x}$.

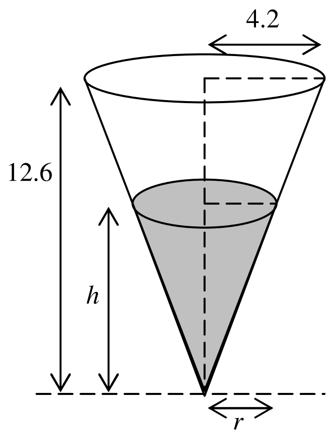
The chain rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$.

Given the velocity of the shadow $= \dfrac{dx}{dt} = 1.5$ at $x = 2.0$,

the vertical velocity $= \dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$

$= -1.76e^{-0.8(2.0)} \times 1.5 = -0.53$ m s^{-1}. The negative sign indicates downward motion.

Example 5 Water is poured into a conical cup at a rate of 5.0 cm^3 per second. The cup is 12.6 cm tall and the radius of the rim of the cup is 4.2 cm. How fast does the water level rise when the depth of water is 8.0 cm?



Use similar triangles to find the relationship between r and h.

$\dfrac{r}{4.2} = \dfrac{h}{12.6}$, $\therefore r = \dfrac{h}{3}$

Volume of a cone $= \dfrac{1}{3}\pi R^2 H$, $\therefore$ volume of water $= V = \dfrac{1}{3}\pi r^2 h$.

Substitute $r = \dfrac{h}{3}$ into the last expression, $V(h) = \dfrac{\pi h^3}{27}$, and

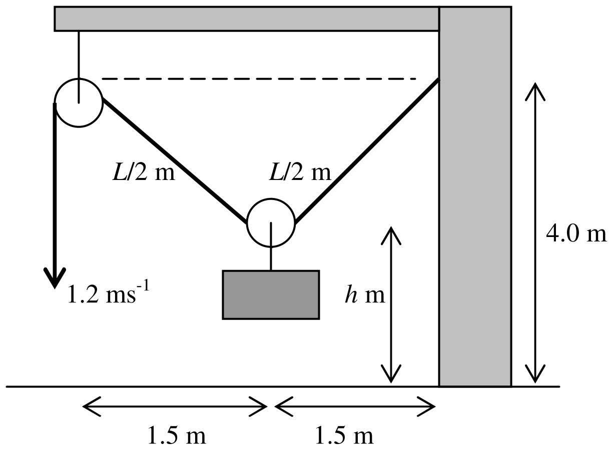
$\dfrac{dV}{dh} = \dfrac{\pi h^2}{9}$.

The chain rule: $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$, $\therefore\ \dfrac{dh}{dt} = \dfrac{dV}{dt} \Big/ \dfrac{dV}{dh}$.

Given $\dfrac{dV}{dt} = 5.0$ (constant), at $h = 8.0$, the rate of change of

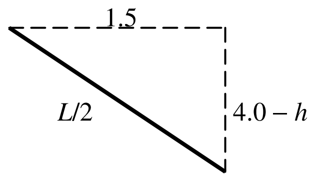
depth $= \dfrac{dh}{dt} = \dfrac{5.0}{\dfrac{\pi(8.0^2)}{9}} = 0.22$ cm.

Example 6 A load is lifted upwards by a person pulling a rope at 1.2 m s^{-1}. See the diagram below. How fast does the load rise when the lower pulley is 2.0 m above the ground?



Let L m be the length of the rope from the upper pulley through the lower pulley to the wall and h m be the height of the lower pulley above the ground. Assume zero size for the pulleys.

Use Pythagoras Theorem to find the relationship between L and h.



$\left(\dfrac{L}{2}\right)^2 = 1.5^2 + (4-h)^2$, $L = 2\sqrt{2.25 + (4-h)^2}$,

$\dfrac{dL}{dh} = \dfrac{1}{\sqrt{2.25 + (4-h)^2}} \times (-2(4-h)) = \dfrac{-2(4-h)}{\sqrt{2.25+(4-h)^2}}$.

The chain rule: $\dfrac{dL}{dt} = \dfrac{dL}{dh} \times \dfrac{dh}{dt}$, $\therefore\ \dfrac{dh}{dt} = \dfrac{dL}{dt} \Big/ \dfrac{dL}{dh}$.

Given $\dfrac{dL}{dt} = -1.2$ (constant, negative because L is decreasing),

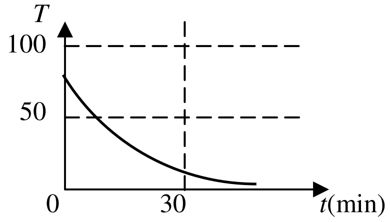
when $h = 2.0$, the rising speed of the load

$= \dfrac{dh}{dt} = \dfrac{-1.2}{\dfrac{-2(4-2)}{\sqrt{2.25+(4-2)^2}}} = 0.75$ m s^{-1}.

Exercise: Next page

1. The position x (in metres) of a particle moving in a straight line is given by $x = t^2 - 8t + 18$ at time t (in seconds). Find the (i) average velocity, i.e. average rate of change of x with respect to t over the interval $[4,5]$ and (ii) instantaneous velocity, i.e. instantaneous rate of change of x with respect to t, at $t = 5$.

2. The graph shows the temperature T (in °C) of boiling water decreases when the burner is turned off at $t = 0$. Estimate (i) the average rate of change in temperature in the first 30 minutes and (ii) the rate of change in temperature at $t = 30$ min.



3. The volume V (in litres) of water remaining in a tank after draining for t minutes is given by $V(t) = 50000\left(1 - \frac{t}{60}\right)^2$. Find the rate at which the water is draining after 30 min.

4. A 4-metre ladder leans against a vertical wall. If the bottom of the ladder slides away from the wall at 0.3 m s^{-1}, find the speed of the top of the ladder sliding down the wall when the bottom of the ladder is 2 m from the wall.

5. Refer to the ladder in Q4. The sliding ladder makes an angle θ with the vertical wall at time t. Find the rate of increase of θ (in ° s^{-1}) when the bottom of the ladder is 2 m from the wall.

6. A spherical balloon is inflated at 80 cm^3 s^{-1}. How fast is the radius r (in cm) increasing when $r = 20$?

7. Refer to the balloon in Q6. How fast is the surface area A (in cm^2) increasing when $r = 20$?

8. Two cars move away from the intersection of two perpendicular straight roads. Car A travels at 60 km h^{-1} and car B at 80 km h^{-1}. If both cars are at the intersection initially, at what rate are they moving apart after 6 min?

9. Refer to the two cars in Q8. At what rate are the two cars moving apart after 6 min if initially car B is at the intersection and car A is 3 km from the intersection?

10. Refer to the two cars in Q8. If both cars are at the intersection initially, at what rate are they moving apart when they are 2 km from each other?

11. The volume of a cube increases at 0.5 cm^3 s^{-1}. How fast does the surface area increase when the length of its edge is 20 cm?

Numerical, algebraic and worded answers.

1. (i) 9 m s^{-1} (ii) 2 m s^{-1}
2. (i) −2.5 °C min^{-1}
   (ii) −0.9 °C min^{-1}
3. 833.3 L min^{-1}
4. 0.1732 m s^{-1}
5. 4.96 ° s^{-1}
6. 0.016 cm s^{-1}
7. 8 cm^2 s^{-1}
8. 100 km h^{-1}
9. 98 km h^{-1}
10. 100 km h^{-1}
11. 0.1 cm^2 s^{-1}