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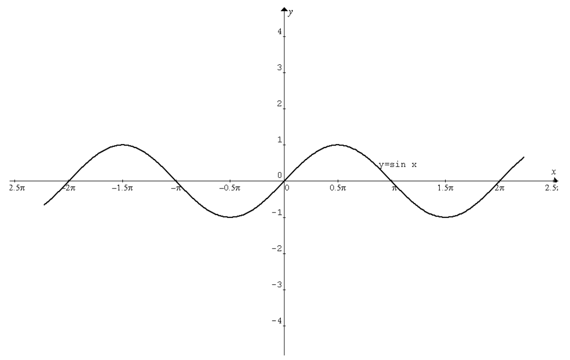
Free maths self-help revision

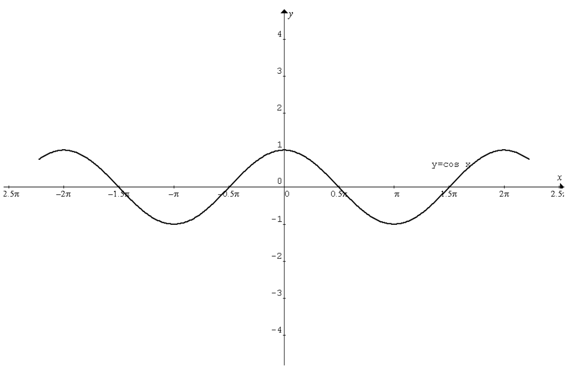
Circular (trigonometric) functions $y = \sin x$, $y = \cos x$

Both $\sin x$ *and* $\cos x$ *are periodic functions. Each repeats itself every interval of* 2π*, i.e. each has symmetry property under a horizontal translation of* 2π*, e.g.* $\sin(x \pm 2\pi) = \sin x$, $\cos(x \pm 2\pi) = \cos x$.

Other symmetry properties:
For $\cos x$*, under a reflection in the y-axis,* $\cos(-x) = \cos x$.
For $\sin x$*, under a reflection in the y-axis and a horizontal translation of* π, $\sin(\pi - x) = \sin x$*; under reflections in both axes,* $-\sin(-x) = \sin x$.

The value of each function fluctuates between −1 and 1. The amplitude of each is 1.
Domain: R; range: $[-1,1]$.





Example 1 Consider the graph of the sine function, $y = \sin x$. Name two ways to change the equation to obtain the graph of the cosine function, $y = \cos x$.

$y = \sin x \rightarrow y = \sin\left(x + \frac{\pi}{2}\right)$ which is $y = \cos x$

$y = \sin\left(x + \frac{\pi}{2}\right)$ is a translation of $y = \sin x$ to the **left** by $\frac{\pi}{2}$.

$y = \sin x \rightarrow y = -\sin\left(x - \frac{\pi}{2}\right)$ which is $y = \cos x$

$y = -\sin\left(x - \frac{\pi}{2}\right)$ is a reflection of $y = \sin x$ in the x-axis and then a translation to the **right** by $\frac{\pi}{2}$.

Example 2 Write down the equation of a cosine function which has an amplitude of 3 and a period of 4π.

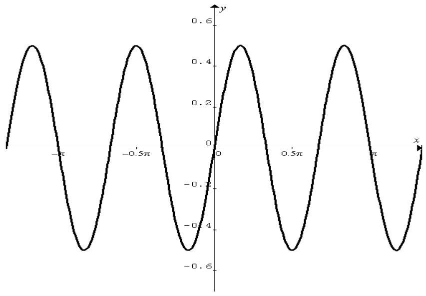
$y = \cos x \rightarrow y = 3\cos\frac{x}{2}$

The 3 in front of cos stretches $y = \cos x$ vertically by a factor of 3, i.e. the amplitude is 3 times the original amplitude of 1, and the 2 under x stretches $y = \cos x$ horizontally by a factor 2, i.e. the period is 2 times the original period of 2π.

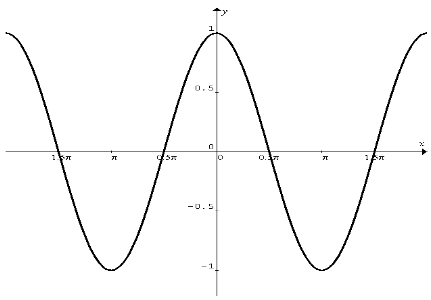
Given the equation of a sine or cosine function, $y = A\sin(nx - \varepsilon)$ *or* $y = A\cos(nx - \varepsilon)$, $|A|$ *is the amplitude, and the period is given by* $T = \frac{2\pi}{|n|}$.
Amplitudes and periods always have positive values.

Example 3 Sketch the graph of $y = \frac{1}{2}\sin(3x)$.

The sine graph has an amplitude of $\frac{1}{2}$ and a period of $\frac{2\pi}{3}$.



Example 4 Sketch the graph of $y = \cos(-x)$.



The graph of $y = \cos(-x)$ is the reflection of $y = \cos x$ in the y-axis and it is the same graph as $y = \cos x$ . .: $\cos(-x) = \cos x$ as pointed out earlier.

Example 5 Use symmetry properties of the sine function to write down 2 positive and 2 negative exact values of a such that $\sin a = \sin\frac{\pi}{5}$.

$\pi - \frac{\pi}{5} = \frac{4\pi}{5}$, $2\pi + \frac{\pi}{5} = \frac{11\pi}{5}$

$-\pi - \frac{\pi}{5} = -\frac{6\pi}{5}$, $-2\pi + \frac{\pi}{5} = -\frac{9\pi}{5}$

Example 6 Use symmetry properties of the cosine function to write down 2 positive and 2 negative exact values of b such that $\cos b = \cos\frac{\pi}{7}$.

$-\frac{\pi}{7}$, $-2\pi + \frac{\pi}{7} = -\frac{13\pi}{7}$

$2\pi - \frac{\pi}{7} = \frac{13\pi}{7}$, $2\pi + \frac{\pi}{7} = \frac{15\pi}{7}$

Circular (trigonometric) function $y = \tan x$

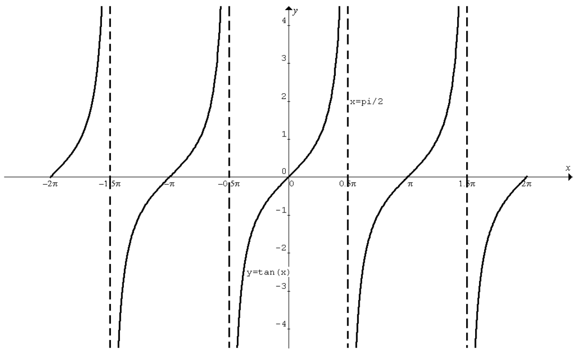
The function $\tan x$ *is also a periodic function. It repeats itself every interval of* π, *i.e. it has symmetry property under a horizontal translation of* π, $\tan(x \pm \pi) = \tan x$.
It also has symmetry property under reflections in both axes, $-\tan(-x) = \tan x$.
The term amplitude is not applicable to $y = \tan x$.

The function is undefined at $x = \pm\left(n - \frac{1}{2}\right)\pi$ *for* $n \in J^+$ *(set of positive integers).*

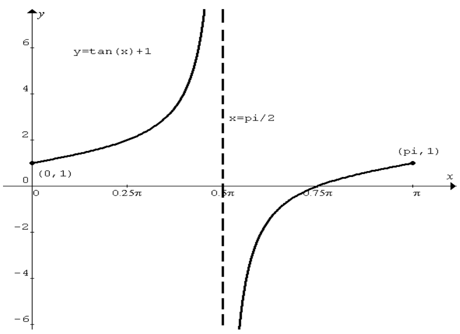
It shows asymptotic behaviour: As $x \to \pm\left(n - \frac{1}{2}\right)\pi$, $\tan x \to \pm\infty$

The vertical asymptotes are $x = \pm\left(n - \frac{1}{2}\right)\pi$.

Domain: $\left\{x : x \neq \pm\left(n - \frac{1}{2}\right)\pi, n \in J^+\right\}$*; range: R.*



Example 7 Sketch the graph of $y = \tan x + 1$, $x \in [0, \pi]$ and $x \neq \frac{\pi}{2}$.



It has the same shape as $y = \tan x$ but the pattern is translated upwards by 1 unit.

It has a vertical asymptote $x = \frac{\pi}{2}$.

x-intercept: $\left(\frac{3\pi}{4}, 0\right)$, y-intercept: $(0,1)$.

The graph shows one cycle of the function.

The period of a tangent function of the form $\tan(nx)$ *or* $\tan(nx \pm b)$ *is given by* $T = \frac{\pi}{|n|}$.

Example 8 Find the period of $\tan\left(\frac{2x-1}{3}\right)$.

$\tan\left(\frac{2x-1}{3}\right) = \tan\left(\frac{2}{3}x - \frac{1}{3}\right)$, .: $T = \frac{\pi}{\frac{2}{3}} = \frac{3\pi}{2}$

Example 9 Use symmetry properties of the tangent function to write down 2 positive and 2 negative exact values of c such that $\tan c = \tan\frac{\pi}{9}$.

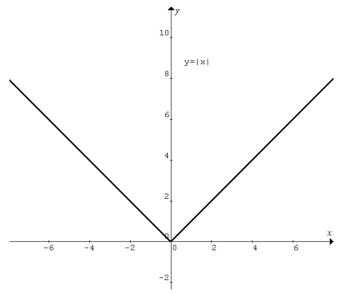
$-2\pi + \frac{\pi}{9} = -\frac{17\pi}{9}$, $-\pi + \frac{\pi}{9} = -\frac{8\pi}{9}$, $\pi + \frac{\pi}{9} = \frac{10\pi}{9}$, $2\pi + \frac{\pi}{9} = \frac{19\pi}{9}$

Modulus function $y = |x|$

The function, modulus of x, can be defined as $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

or $|x| = \sqrt{x^2}$.

It has symmetry property under reflection in the y-axis, $|x| = |-x|$, *i.e.* $|x|$ *and* $|-x|$ *are the same function, and the line* $x = 0$ *is the axis of symmetry. Its vertex is* $(0,0)$. *Domain: R; range:* $[0, \infty)$.



Example 10 Find the angle between the 2 branches of $y = |x|$.

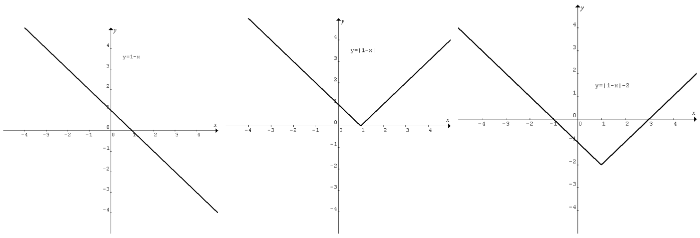
Since $y = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$, and the gradients of the 2 branches are -1 for $x < 0$ and 1 for $x \geq 0$, $-1 \times 1 = -1$, .: the 2 branches are perpendicular and make a $90°$ angle.

Example 11 Are the 2 branches of $y = |2x|$ perpendicular?

No, because the product of the gradients $\neq -1$.

Example 12 Sketch the graphs of $y = |1 - x|$ and $y = |1 - x| - 2$ without CAS.

Step 1. Sketch $y = 1 - x$ Step 2. Reflect the section below the x-axis to obtain $y = |1 - x|$ Step 3. Translate $y = |1 - x|$ downwards by 2 units to obtain $y = |1 - x| - 2$.



Example 13 Find the axes intercepts of the graph of $y = |1 - x| - 2$ without CAS.

y-intercept: Let $x = 0$, $y = |1 - 0| - 2 = 1 - 2 = -1$, $(0,-1)$

x-intercepts: Let $y = 0$, .: $|1 - x| - 2 = 0$, $|1 - x| = 2$
.: $1 - x = 2$ or $1 - x = -2$, .: $x = -1$ or 3, $(-1,0)$, $(3,0)$.

Exercise: Next page

Q1 Consider the graph of the cosine function, $y = \cos x$. Name a change to the equation to obtain the graph of the sine function, $y = \sin x$.

Q2 Write down the equation of a sine function which has an amplitude of 5 and a period of π.

Q3 Sketch the graph of $y = 3\cos\left(\frac{x}{2}\right)$.

Q4 Sketch the graph of $y = \sin(-x)$ and compare it with the graph of $y = \sin x$.

Q5 Use symmetry properties of the cosine function to write down 2 positive and 2 negative exact values of b such that $\cos b = \cos\frac{2\pi}{3}$.

Q6 Sketch the graph of $y = -\tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Q7 Find the period of (a) $\cos\frac{\pi x}{3}$ (b) $\tan\left(\frac{\pi - x}{2}\right)$.

Q8 Use symmetry properties of the tangent function to write down 2 positive and 2 negative exact values of a such that $\tan a = \tan\frac{2}{\pi}$.

Q9 Write $f(x) = \left|x + \frac{1}{2}\right|$ as a *hybrid* function.

Q10 Find the intercepts of $y = 1 - |x + 2|$ with the axes.

Q11 Sketch the graph of $y = 1 - |x + 2|$.

Numerical, algebraic and worded answers:

1. Translate $y = \cos x$ to the right by $\pi/2$. 2. $y = 5\sin(2x)$

4. $\sin(-x)$ is the reflection of $\sin x$ in the y-axis.

5. $-4\pi/3$ $-2\pi/3$, $4\pi/3$, $8\pi/3$ 7a. $T = 6$ 7b. 4π

8. $-2(\pi^2 - 1)/\pi$, $-(\pi^2 - 2)/\pi$, $(\pi^2 + 2)/\pi$, $2(\pi^2 + 1)/\pi$

9. $f(x) = \begin{cases} -(x + 1/2) & x < -1/2 \\ x + 1/2 & x \geq -1/2 \end{cases}$ 10. $(0,-1)$, $(-3,0)$, $(-1,0)$