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Matrices for transformations

The table below shows the matrices for some transformations:

Transformation	Matrix
Reflection in the x-axis	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$
Reflection in the y-axis	$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$
Dilation by factor n from the y-axis	$\begin{bmatrix}n & 0\\ 0 & 1\end{bmatrix}$
Dilation by factor n from the x-axis	$\begin{bmatrix}1 & 0\\ 0 & n\end{bmatrix}$
Horizontal translation by b units	$\begin{bmatrix}b\\ 0\end{bmatrix}$
Vertical translation by c units	$\begin{bmatrix}0\\ c\end{bmatrix}$

These matrices are applied to a point $\begin{bmatrix}x\\ y\end{bmatrix}$ *of the function.*

$\begin{bmatrix}a_{11} & 0\\ 0 & a_{22}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$ *for reflections and dilations;* $\begin{bmatrix}b\\ c\end{bmatrix}+\begin{bmatrix}x\\ y\end{bmatrix}$ *for translations.*

Example 1 $(3,-2)$ is reflected in the x-axis and dilated from the y-axis by a factor of $\frac{1}{2}$. Find (a) the matrix for the combined transformations (b) the matrix for the combined transformations in reverse order (c) the coordinates of the image of $(3,-2)$.

(a) $\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & -1\end{bmatrix}$

(b) $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & -1\end{bmatrix}$

(c) $\begin{bmatrix}\frac{1}{2} & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}3\\ -2\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\ 2\end{bmatrix}$, the image is $\left(\frac{3}{2},2\right)$.

Note: For dilations and reflections the order of operation of the transformations is not important as shown in (a) and (b).

Example 2 $(-1,5)$ is (a) translated by 2 units in the positive x direction and then reflected in the y-axis (b) reflected in the y-axis and then translated by 2 units in the positive x direction. Find the image of $(-1,5)$ in each case.

(a) $\begin{bmatrix}2\\ 0\end{bmatrix}+\begin{bmatrix}-1\\ 5\end{bmatrix}=\begin{bmatrix}1\\ 5\end{bmatrix}$, $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 5\end{bmatrix}=\begin{bmatrix}-1\\ 5\end{bmatrix}$, image $(-1,5)$

(b) $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1\\ 5\end{bmatrix}=\begin{bmatrix}1\\ 5\end{bmatrix}$, $\begin{bmatrix}2\\ 0\end{bmatrix}+\begin{bmatrix}1\\ 5\end{bmatrix}=\begin{bmatrix}3\\ 5\end{bmatrix}$, image $(3,5)$

The two combinations of transformations are:

$T\left(\begin{bmatrix}x\\ y\end{bmatrix}\right)=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}2\\ 0\end{bmatrix}+\begin{bmatrix}x\\ y\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}x\\ y\end{bmatrix}\right)=\begin{bmatrix}2\\ 0\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$ respectively.

Note: For translations together with dilations and/or reflections the order of operation must be adhered to as seen in (a) and (b).

Example 3 Find the equation of the image of $y=(x+1)^3+2$ under a reflection in the y-axis and a dilation from the y-axis by a factor of $\frac{1}{3}$.

The transformation matrix is $\begin{bmatrix}\frac{1}{3} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{3} & 0\\ 0 & 1\end{bmatrix}$.

Let $\begin{bmatrix}x'\\ y'\end{bmatrix}$ be the image point of a general point $\begin{bmatrix}x\\ y\end{bmatrix}$ of the original equation.

$\begin{bmatrix}x'\\ y'\end{bmatrix}=\begin{bmatrix}-\frac{1}{3} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}x\\ y\end{bmatrix}$, .: $x'=-\frac{1}{3}x$ and $y'=y$

.: $x=-3x'$ and $y=y'$

Equation of the image: $y'=(-3x'+1)^3+2$, .: $y=(-3x+1)^3+2$

Alternatively, find the inverse of $\begin{bmatrix}-\frac{1}{3} & 0\\ 0 & 1\end{bmatrix}$

$\begin{bmatrix}-\frac{1}{3} & 0\\ 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}-3 & 0\\ 0 & 1\end{bmatrix}$, .: $\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x'\\ y'\end{bmatrix}=\begin{bmatrix}-3x'\\ y'\end{bmatrix}$

.: $x=-3x'$ and $y=y'$ etc.

Example 4 Use matrix method to do Example 12 in the previous self-help revision.

Let $\begin{bmatrix}x'\\ y'\end{bmatrix}$ be the image point of a general point $\begin{bmatrix}x\\ y\end{bmatrix}$ of the original equation.

Transformations (2) and (3) form a single matrix, $\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}$.

$\begin{bmatrix}x'\\ y'\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}0\\ -2\end{bmatrix}+\begin{bmatrix}x\\ y\end{bmatrix}\right)=\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y-2\end{bmatrix}=\begin{bmatrix}-2x\\ y-2\end{bmatrix}$

.: $x'=-2x$ and $y'=y-2$, .: $x=-\frac{1}{2}x'$ and $y=y'+2$

Equation of the transformed function (image) of $y=1-\sqrt{2-x}$:

$y'+2=1-\sqrt{2+\frac{x'}{2}}$, .: $y=-1-\sqrt{2+\frac{x}{2}}$

Example 5 (VCAA 2013 Exam 2)

A transformation $T:R^2\to R^2$, $T\left(\begin{bmatrix}x\\ y\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}5\\ 0\end{bmatrix}$ maps the graph of a function f to the graph of $y=x^2$, $x\in R$. Find the rule of f.

$\begin{bmatrix}x'\\ y'\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}5\\ 0\end{bmatrix}=\begin{bmatrix}x+5\\ -y\end{bmatrix}$, .: $x'=x+5$ and $y'=-y$

The equation of the image is $y=x^2$, i.e. $y'=x'^2$

.: equation of the original function is $-y=(x+5)^2$,

$y=-(x+5)^2$, .: $f(x)=-(x+5)^2$

Alternative method:

T is a sequence of transformations on f: Reflection in the x-axis followed by a translation of 5 units to the right. Find f by back tracking the sequence of transformations on the image, i.e. translation to the left by 5 units and then reflection in the x-axis.

Image $y=x^2\to y=(x+5)^2\to -y=(x+5)^2$

.: $f(x)=-(x+5)^2$

Matrix methods in solving simultaneous linear equations

Example 6 The graph of $y = ax^4 + bx^3 + cx^2 + dx + e$ passes through $(-3,0)$, $(-1,1)$, $(1,1)$, $(2,0)$ and $(-2,0)$. Find the values of *a*, *b*, *c*, *d* and *e*.

Substitute the coordinates of each point into the equation to obtain five simultaneous equations in *a*, *b*, *c*, *d* and *e*.

$$\begin{aligned} &81a - 27b + 9c - 3d + e = 0 \\ &a - b + c - d + e = 1 \\ &a + b + c + d + e = 1 \\ &16a + 8b + 4c + 2d + e = 0 \\ &16a - 8b + 4c - 2d + e = 0 \end{aligned}$$

In matrix form:

$$\begin{bmatrix} 81 & -27 & 9 & -3 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 81 & -27 & 9 & -3 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 16 & -8 & 4 & -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{24} \\ 0 \\ -\frac{13}{24} \\ 0 \\ \frac{3}{2} \end{bmatrix}$$

$\therefore\ a = \frac{1}{24}$, $b = 0$, $c = -\frac{13}{24}$, $d = 0$ and $e = \frac{3}{2}$

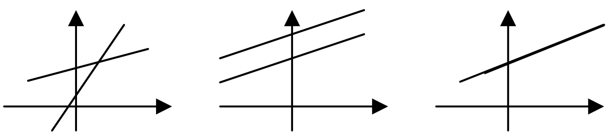
Solving simultaneous linear equations with 2 unknowns

A practical case is finding the intersection of two straight lines in the Cartesian plane. Each equation represents a straight line. There are three possibilities:

(1) The two lines intersect at a point. ∴ the two equations have a solution. This occurs when the two lines have different gradients.

(2) The two lines are parallel and do not intersect. ∴ the two equations have the same gradient but different y-intercepts.

(3) The two lines overlap each other, i.e. there is an infinite number of intersections. ∴ the two equations have infinitely many solutions. This occurs when the two lines have the same gradient and the same y-intercept, i.e. the two equations are the same.



Example 7 Solve the simultaneous equations by matrix method. $2y - x + 1 = 0$, $5 - 2x - 3y = 0$

Rewrite the equations:

$x - 2y = 1$

$2x + 3y = 5$

Change to matrix form: $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \approx \begin{bmatrix} 1.8571 \\ 0.4286 \end{bmatrix}$

Example 8 Show that the simultaneous equations $2x + 3y = 5$ and $0.1x + 0.15y = 0.25$ cannot be solved by matrix method.

$\begin{bmatrix} 2 & 3 \\ 0.1 & 0.15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0.25 \end{bmatrix}$, the inverse of $\begin{bmatrix} 2 & 3 \\ 0.1 & 0.15 \end{bmatrix}$ does not exist, ∴ cannot be solved by matrix method.

Note: Matrix method is not applicable when $ad - bc = 0$ where a, b, c and d are the elements of the square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of the simultaneous equations. $ad - bc$ is called the determinant of the square matrix. When $ad - bc = 0$, either there are infinitely many solutions to the simultaneous equations or none at all.

Example 9 Find the solutions to the simultaneous equations $2x + 3y = 5$ and $0.1x + 0.15y = 0.25$.

20 times the second equation will give the same equation as the first one, ∴ the two given equations represent two overlapping lines, ∴ there are infinitely many solutions.
The solution set to the two simultaneous equations is $\{(x, y): 2x + 3y = 5\}$, i.e. the set of points on the line $2x + 3y = 5$

Example 10 Explain why there are no solutions to the simultaneous equations $2x + 3y = 5$ and $\frac{2}{3}x + y = 2$.

There are no solutions because they represent two parallel lines (same gradient) with different *y*-intercept.

In matrix form, $\begin{bmatrix} 2 & 3 \\ \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $ad - bc = 0$.

Example 11 (VCAA 2011 Exam 1) Consider the simultaneous linear equations $kx - 3y = k + 3$ and $4x + (k + 7)y = 1$ where *k* is a real constant. (a) Find the value of *k* for which there are infinitely many solutions. (b) Find the value of *k* for which there are no solutions. (c) Find the values of *k* for which there is a unique solution.

(a) Express the equations in $y = mx + c$ form.

$y = \frac{k}{3}x - \frac{k+3}{3}$, $y = -\frac{4}{k+7}x + \frac{1}{k+7}$

Infinitely many solutions: Same gradient AND same *y*-intercept.

$\frac{k}{3} = -\frac{4}{k+7}$ AND $-\frac{k+3}{3} = \frac{1}{k+7}$

$k(k+7) + 12 = 0$ AND $(k+3)(k+7) + 3 = 0$

$k^2 + 7k + 12 = 0$ AND $k^2 + 10k + 24 = 0$

$\therefore\ k = -3$ or -4 AND $k = -6$ or -4

$\therefore\ k = -4$

No solutions: Same gradient AND different *y*-intercepts.

$\frac{k}{3} = -\frac{4}{k+7}$ AND $-\frac{k+3}{3} \neq \frac{1}{k+7}$

$k^2 + 7k + 12 = 0$ AND $k^2 + 10k + 24 \neq 0$

$\therefore\ k = -3$ or $k = -4$ AND $k \neq -6$ or $k \neq -4$

$\therefore\ k = -3$

A unique solution: Different gradients.

$\frac{k}{3} \neq -\frac{4}{k+7}$, $k^2 + 7k + 12 \neq 0$, $k \neq -3$, $k \neq -4$,

i.e. $k \in R \setminus \{-4, -3\}$

Example 12 For the simultaneous linear equations $2x - ky = 1$ and $2kx - 2y = k + 1$ where $k \neq 0$ is a real constant, show that it is not possible to have infinitely many solutions.

Express the equations in $y = mx + c$ form.

$y = \frac{2}{k}x - \frac{1}{k}$, $y = kx - \frac{k+1}{2}$

Infinitely many solutions: Same gradient AND same *y*-intercept.

$\therefore\ \frac{2}{k} = k$ and $\frac{1}{k} = \frac{k+1}{2}$, no real $k \neq 0$ can satisfy both equations.

∴ not possible to have infinitely many solutions.

Exercise: Next page

Q1 $(1,-3)$ is reflected in the y-axis and dilated from the x-axis by a factor of $\frac{1}{2}$. Find (a) the matrix for the combined transformations (b) the coordinates of the image of $(1,-3)$.

Q2 $(-3,1)$ is translated by 2 units in the positive x direction and then reflected in the y-axis. Find (a) the matrix representation of the combined transformations (b) the coordinates of the image of $(-3,1)$.

Q3 Refer to Q2. If the transformations are carried out in reverse order, find (a) the matrix representation of the combined transformations (b) the coordinates of the image of $(-3,1)$.

Q4 Use matrix method to find the equation of the image of $y=\sqrt{2-x}+2$ under a reflection in the y-axis and a dilation from the y-axis by a factor of 2.

Q5 Use matrix method to find the equation of the transformed function when $y=\frac{1}{x+1}$ undergoes (1) a downward translation by 2 units, (2) a horizontal dilation by a factor of 2 and (3) a reflection in the y-axis.

Q6 $(-1,1)$, $(2,0)$, $(1,0)$ and $(-2,-1)$ are points of the cubic polynomial $y=ax^3+bx^2+cx+d$. Find the values of a, b, c and d by matrix method.

Q7 Solve the simultaneous equations $2x-1+y=0$ and $2y-3x=1$ by matrix method.

Q8 Find the coordinates of the intersection of $2x+3y=5$ and $x-2y=3$ by matrix method.

Q9 Find the solutions to the simultaneous equations $x+2y=1$ and $0.5x+y=0.5$.

Q10 Consider the simultaneous linear equations $mx-2y=2$ and $4x+3y=n$ where m and n are real constants. (a) Find the values of m and n for which there are infinitely many solutions. (b) Find the values of m and n for which there are no solutions. (c) Find the values of m and n for which there is a unique solution.

Q11 Consider the simultaneous linear equations $(k-1)x-\frac{1}{k+1}y=\frac{1}{k+1}$ and $(k-1)x-y=\frac{1}{k}$ where k is a real constant. Show that there is at least one solution.

Numerical, algebraic and worded answers:

1. $\begin{bmatrix}-1 & 0\\ 0 & 0.5\end{bmatrix}$, $(-1,-1.5)$ 2. $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}-3\\ 1\end{bmatrix}+\begin{bmatrix}2\\ 0\end{bmatrix}\right)$, $(1,1)$ 7. $x=\frac{1}{7}$, $y=\frac{5}{7}$

3. $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-3\\ 1\end{bmatrix}+\begin{bmatrix}2\\ 0\end{bmatrix}$, $(5,1)$ 4. $y=\sqrt{2+\frac{x}{2}}+2$ 5. $y=\frac{2}{2-x}-2$

6. $a=\frac{1}{4}$, $b=-\frac{1}{3}$, $c=-\frac{3}{4}$, $d=\frac{5}{6}$ 8. $x=\frac{19}{7}$, $y=-\frac{1}{7}$

9. $\{(x,y): x+2y=1\}$ 10a. $m=-\frac{8}{3}$, $n=-3$ 10b. $m=-\frac{8}{3}$, $n\neq -3$

10c. $m\neq -\frac{8}{3}$, any real n