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Polynomial functions

A polynomial function $P(x)$ *is a linear combination of power functions* x^n *, where* $n \in \{0, 1, 2.....\}$

Examples are:

$P(x) = 2x - 5$ *a linear function*

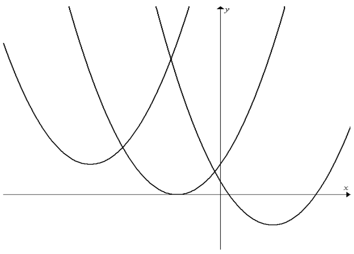
$P(x) = -3x^2 + x + 2$ *a quadratic function*

$P(x) = 0.2x^3 - \dfrac{x^2}{3} + (\sqrt{5})x - \pi$ *a cubic function*

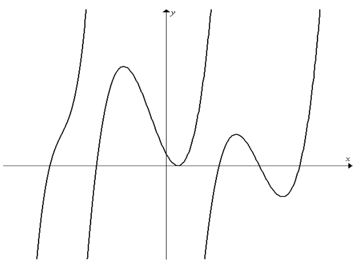
$P(x) = -x^4 + (\sqrt[3]{4})x - e^2$ *a quartic function*

Some polynomial functions can be changed to factorised form. Linear factors give the x-intercepts. Some polynomial functions do not have linear factors. Hence not all polynomial functions have x-intercepts.

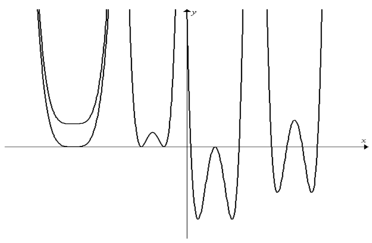
A quadratic function may have 0, 1 or 2 distinct linear factors, hence 0, 1 or 2 x-intercepts.



A cubic function may have 1, 2 or 3 distinct linear factors, hence 1, 2 or 3 x-intercepts.



A quartic function may have 0, 1, 2, 3 or 4 distinct linear factors, hence 0, 1, 2, 3 or 4 x-intercepts.



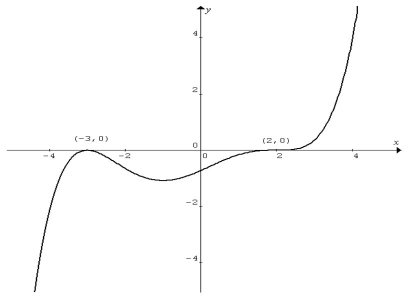
If the power of a linear factor in a polynomial is even, then the corresponding x-intercept is a turning point.
If the power of a linear factor in a polynomial is odd and greater than 1, then the corresponding x-intercept is a stationary point of inflection.

For $y = a(x-b)^2(x-c)^3(x-d)^4(x-e)^5(x^2+f)$,
the x-intercepts at $x = b$ *and* $x = d$ *are turning points; and the x-intercepts at* $x = c$ *and* $x = e$ *are stationary points of inflection. The factor* $x^2 + f$ *(* $f > 0$ *) is not linear and* $\therefore$ *does not correspond to an x-intercept.*

If a is a positive (negative) value, the graph of a polynomial function heads upwards (downwards) in the positive x-direction.

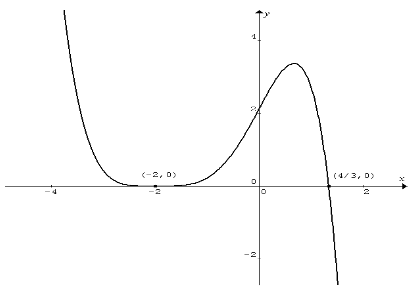
Example 1 Sketch $y = \dfrac{1}{100}(x+3)^2(x-2)^3$

The x-intercept at $x = -3$ is a turning point; at $x = 2$ the x-intercept is a stationary point of inflection.

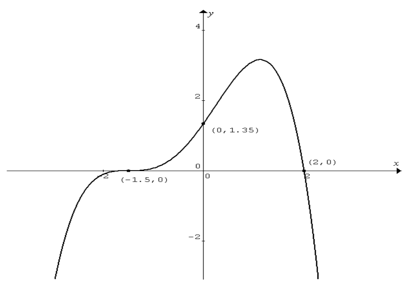


Example 2 Sketch $y = \dfrac{1}{30}(4-3x)(x+2)^4$

Express the function as $y = -\dfrac{1}{10}\left(x - \dfrac{4}{3}\right)(x+2)^4$. The function crosses the x-axis at $x = \dfrac{4}{3}$; it touches the x-axis at $x = -2$.



Example 3 Find the equation of the quartic function shown below.



At $x = -\dfrac{3}{2}$ the function has an x-intercept that is a stationary point of inflection; at $x = 2$ the function crosses the x-axis.

Hence $y = a\left(x + \dfrac{3}{2}\right)^3 (x-2)$, where $|a|$ is the vertical dilation factor to be determined using further information, in this case, the y-intercept $(0, 1.35)$.

$1.35 = a\left(\dfrac{3}{2}\right)^3(-2)$, $\therefore a = -\dfrac{1}{5}$ $\quad \therefore y = -\dfrac{1}{5}\left(x + \dfrac{3}{2}\right)^3(x-2)$

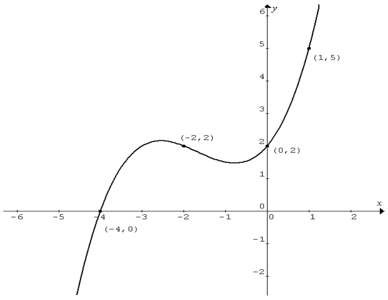
This quartic function can also be expressed as

$y = -\dfrac{1}{5}\left[\dfrac{1}{2}(2x+3)\right]^3(x-2) = -\dfrac{1}{5}\left(\dfrac{1}{2}\right)^3(2x+3)(x-2)$

$= -\dfrac{1}{40}(2x+3)(x-2)$

The − sign corresponds to the graph heading downwards in the positive x-direction.

Example 4 Find the equation of the cubic function shown in the graph below.



The cubic function has only one x-intercept at $x=-4$, and $\therefore$ only one linear factor.
Its equation in factorised form must be $y=a(x+4)(x^2+bx+c)$.

Use the other given points to set up simultaneous equations, then solve for a, b and c.

$(0,2)\rightarrow 2=4ac \qquad \therefore ac=0.5 \qquad (1)$

$(-2,2)\rightarrow 2=2a(4-2b+c) \qquad \therefore 4a-2ab+ac=1 \qquad (2)$

$(1,5)\rightarrow 5=5a(1+b+c) \qquad \therefore a+ab+ac=1 \qquad (3)$

Note: These equations are not simultaneous linear equations. Do not try to solve them by matrix method.

Substitute equation (1) in equations (2) and (3):

$4a-2ab=0.5 \qquad (4)$

$a+ab=0.5 \qquad (5) \quad \times 2$

$2a+2ab=1 \qquad (6)$

Add equations (4) and (6), $6a=1.5$, $\therefore a=0.25 \qquad (7)$
Substitute equation (7) in (5) to obtain $b=1$
Substitute equation (7) in (1) to obtain $c=2$

Hence $y=0.25(x+4)(x^2+x+2)$.

Alternative method: $y=(x+4)(ax^2+bx+c)$

$(0,2)\rightarrow 2=4c \qquad \therefore c=\frac{1}{2} \qquad (1)$

$(-2,2)\rightarrow 2=2(4a-2b+c) \qquad \therefore 4a-2b+c=1 \qquad (2)$

$(1,5)\rightarrow 5=5(a+b+c) \qquad \therefore a+b+c=1 \qquad (3)$

These are simultaneous linear equations. In matrix form:

$$\begin{bmatrix}0&0&1\\4&-2&1\\1&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\1\\1\end{bmatrix},\ \therefore \begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0&0&1\\4&-2&1\\1&1&1\end{bmatrix}^{-1}\begin{bmatrix}\frac{1}{2}\\1\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{2}\end{bmatrix}$$

$$\therefore\ y=(x+4)\left(\frac{1}{4}x^2+\frac{1}{4}x+\frac{1}{2}\right)=\frac{1}{4}(x+4)(x^2+x+2)$$

***All** quadratic functions can be changed to turning point form $y=A(x-b)^2+c$ by completing the square.
The turning point is (b,c).*

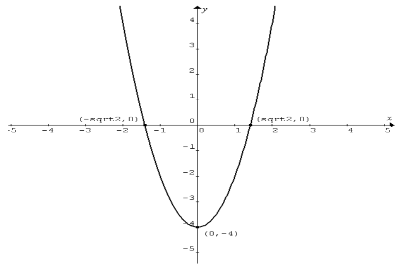
***Some** cubic functions can be expressed in the same form $y=A(x-b)^3+c$.
(b,c) is the stationary point of inflection of the cubic function.*

***Some** quartic functions can also be expressed in the same form $y=A(x-b)^4+c$.
(b,c) is the turning point of the quartic function.*

These forms could be viewed as the transformations of the power functions x^2, x^3 and x^4 respectively.

Example 5 Find the turning point and the x-intercepts of $y=2x^2-4$. Sketch its graph.

The function is in turning point form. The turning point is $(0,-4)$.
Factorise $y=2x^2-4=2(x^2-2)=2(x-\sqrt{2})(x+\sqrt{2})$.
The linear factor $x-\sqrt{2}$ gives x-intercept $(\sqrt{2},0)$ and the other linear factor $x+\sqrt{2}$ gives x-intercept $(-\sqrt{2},0)$.
The y-intercept is obtained by putting $x=0$, $(0,-4)$.



Example 6 Factorise $2(x+1)^3+2$ and sketch $y=2(x+1)^3+2$.

$y=2(x+1)^3+2=2[(x+1)^3+1^3]$

$=2[(x+1)+1][(x+1)^2-(x+1)(1)+(1)^2]$

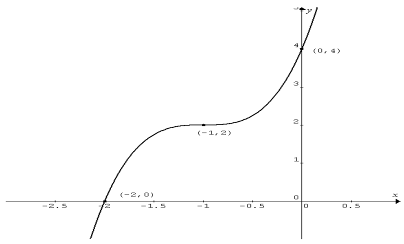
$=2(x+2)(x^2+x+1)$.

There is only one linear factor, $\therefore$ only one x-intercept at $x=-2$.

Note that the x-intercept can also be obtained by letting $y=0$ and solve for x. $2(x+1)^3+2=0$, $2(x+1)^3=-2$, $(x+1)^3=-1$, $x+1=\sqrt[3]{-1}=-1$, $\therefore x=-2$.

Let $x=0$ to obtain $y=4$, $\therefore$ y-intercept is $(0,4)$.

The given function is in stationary inflection point form. The stationary point of inflection is $(-1,2)$.

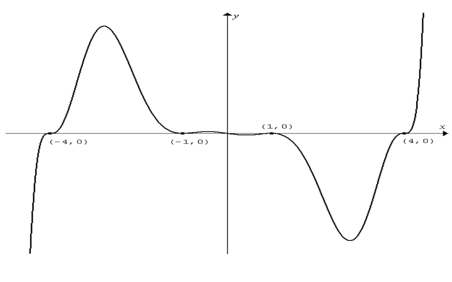


Example 7 An odd polynomial function $P(x)$ has $x=-1$ as a double zero and $x=4$ as a triple zero. What is the least degree possible for such function? Find a polynomial function $P(x)$ with the least degree. Sketch the graph of $P(x)$.

$x=-1$ is a double zero means $(-1,0)$ is a turning point. $x=4$ is a triple zero means $(4,0)$ is a stationary inflection point.
An odd function satisfies $P(-x)=-P(x)$, $\therefore P(x)$ has odd degree.
The least degree possible is 11.

$P(x)=x(x+1)^2(x-1)^2(x-4)^3(x+4)^3$ or

$P(x)=-x(x+1)^2(x-1)^2(x-4)^3(x+4)^3$



Exercise: Next page

Q1 Sketch $y=-2(x+1)(x-1)(x-2)$ without CAS. Show intercepts with the axes. Turning points are not required.

Q2 Sketch $y=\frac{1}{2}(x+3)(x-1)$ without CAS. Find the turning point with no calculus. Find the equation of the parabola which has exactly the same shape as the given one but its vertex is on the *x*-axis.

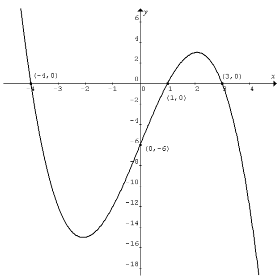
Q3 Sketch $y=-\frac{1}{100}(x-3)^3(x+2)^2$ without CAS.

Q4 Sketch $y=\frac{1}{50}(2-3x)(2-x)^4$ without CAS.

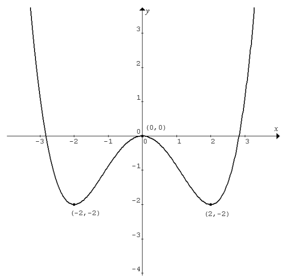
Q5 Find the equation of a quartic function which has a stationary inflection point at $(2,0)$, and $(-3,0)$, $(0,1)$ are the *x* and *y* intercepts.

Q6 Find the equation of a cubic polynomial which passes through the points $(-2,36)$, $(-1,6)$, $(0,-2)$, $(3,-14)$.

Q7 Find the equation of the cubic shown in the graph.



Q8 Find the equation of the quartic shown in the graph.



Q9 Find the turning point and the *x*-intercepts of $y=3x^2-3$. Sketch its graph without CAS.

Q10 Factorise $3(x+2)^3+81$ and sketch $y=3(x+2)^3+81$ without CAS.

Q11 Find an odd polynomial function which has a turning point at $(-2,0)$, and its degree is the lowest possible.

Numerical, algebraic and worded answers:

1. $x=-1,1,2$ $y=-4$ 2. $(-1,-2)$, $y=\frac{1}{2}(x+3)(x-1)+2$

5. $y=-\frac{1}{24}(x+3)(x-2)^3$ 6. $y=-2x^3+5x^2-x-2$

7. $y=-\frac{1}{2}(x+4)(x-1)(x-3)$ 8. $y=\frac{1}{8}(x^2-4)^2-2$

9. $(0,-3)$, $x=\pm 1$ 10. $3(x+5)(x^2+x+7)$

11. $y=x^5-8x^3+16x$