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Graphs of sum and difference of functions

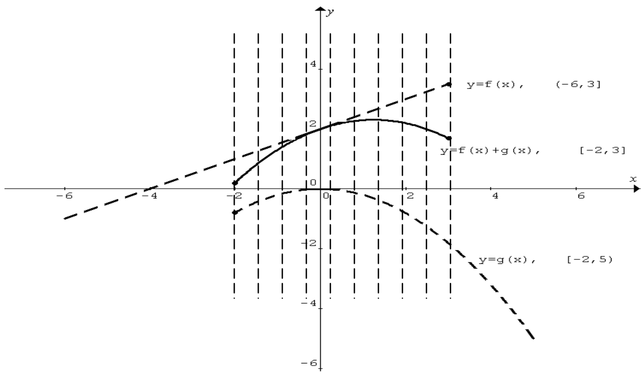
The sum (or difference) of two functions f and g is defined only for $x \in D_f \cap D_g$*, where* D_f *and* D_g *are the domains of f and g respectively.*

Example 1 Given functions *f* and *g* defined by $f(x)=\sqrt{x+1}$ and $g(x)=\log_e(2-x)$ respectively, find the domain of $f+g$.

$f(x)=\sqrt{x+1}$, $\therefore x+1 \ge 0$, $\therefore x \ge -1$, $\therefore D_f=\{x : x \ge -1\}$.
$g(x)=\log_e(2-x)$, $\therefore 2-x>0$, $\therefore x<2$, $\therefore D_g=\{x : x<2\}$.

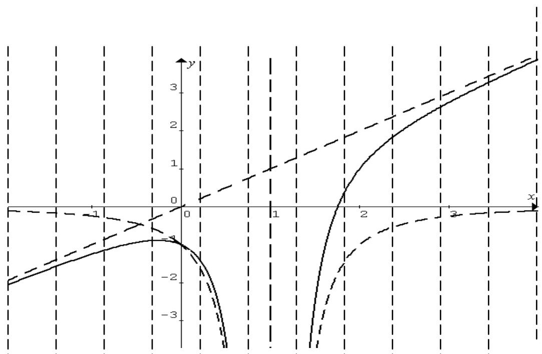
Hence, $D_{f+g}=D_f \cap D_g=\{x : -1 \le x < 2\}$, or $[-1,2)$.

If the graphs of $y=f(x)$ *and* $y=g(x)$ *are given, the graph of* $y=f(x)+g(x)$ *can be sketched by the method of addition of ordinates, i.e. by adding the y-coordinates of the two functions at regular x values in* $D_f \cap D_g$.



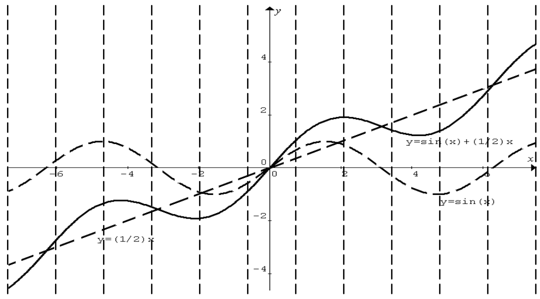
Example 2 Use addition of ordinates to sketch $y=x-\dfrac{1}{(x-1)^2}$.

Sketch $y=x$ and $y=-\dfrac{1}{(x-1)^2}$ on the same axes, then add the *y*-coordinates of the two functions at regular *x*-values.
Note: $y=x-\dfrac{1}{(x-1)^2}$ is undefined at $x=1$, .: domain is $R\setminus\{1\}$.



Example 3 Sketch $y=\sin x+\dfrac{1}{2}x$ by addition of ordinates

Sketch $y=\sin x$ and $y=\dfrac{1}{2}x$ on the same axes, then add the *y*-coordinates of the two functions at regular *x*-values.



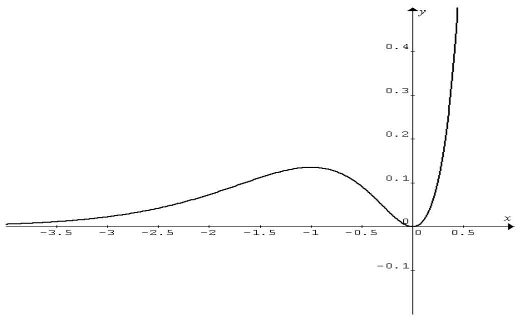
Graph of product of functions

New functions can be generated by + or − of functions. New functions can also be generated by multiplication (or division) of functions. They are called products (or quotients) of functions.

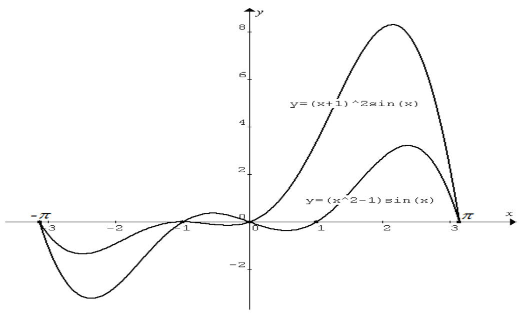
The product (or quotient) of two functions u and v is defined only for $x \in D_u \cap D_v$*, and* $v \ne 0$ *if v is the divisor.*

If the graphs of $y=u(x)$ *and* $y=v(x)$ *are given, then the graph of* $y=u(x)v(x)$ *or* $\left(y=\dfrac{u(x)}{v(x)}\right)$ *can be sketched by ×/÷ the y-coordinate of one function by the y-coordinate of the other at regular x values within* $D_u \cap D_v$.

Example 4 $y=x^2e^{2x}$ is the product of functions $u(x)=x^2$ and $v(x)=e^{2x}$. $D_u=R$ and $D_v=R$, $D_{uv}=D_u \cap D_v=R$.



Example 5 Sketch $y=(x^2-1)\sin x$ and $y=(x+1)^2\sin x$ for $-\pi \le x \le \pi$ on the same axes. Find the intersections without CAS.

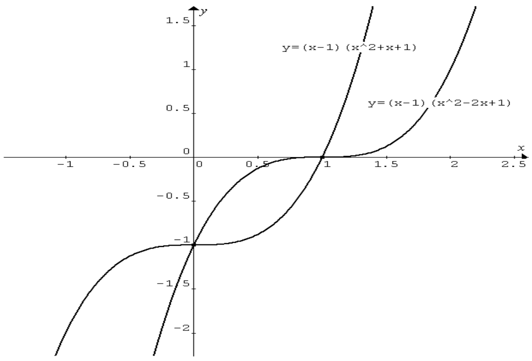


Intersections: Solve the two equations simultaneously.
$(x+1)^2\sin x=(x^2-1)\sin x$
$(x+1)^2\sin x=(x-1)(x+1)\sin x$
Do not cancel common factors from both sides. It will result in loss of solutions and/or inconsistency like $(x+1)=(x-1)$*. Why?*

Correct steps: $(x+1)^2\sin x-(x-1)(x+1)\sin x=0$
$((x+1)-(x-1))(x+1)\sin x=0$, .: $2(x+1)\sin x=0$
.: $x+1=0$ or $\sin x=0$
.: $x=-1$, $-\pi$, 0 or π
The intersections are $(-\pi,0)$, $(-1,0)$, $(0,0)$ and $(\pi,0)$.

Example 6 Sketch $y=(x-1)(x^2+x+1)$ and $y=(x-1)(x^2-2x+1)$ on the same axes. Find the intersections without CAS.

Study the two products of functions carefully before you start sketching. The first one can be expanded to $y=x^3-1$ and the second one factorised to $y=(x-1)^3$.



Intersections: Solve the two equations simultaneously.

$(x-1)(x^2+x+1)=(x-1)(x^2-2x+1)$

Do not cancel common factors from both sides.

$(x-1)(x^2+x+1)-(x-1)(x^2-2x+1)=0$

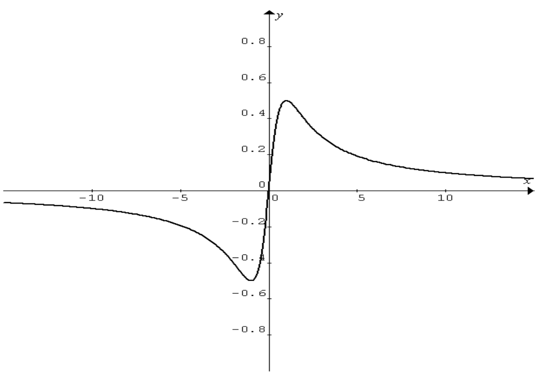
$(x-1)((x^2+x+1)-(x^2-2x+1))=0$, ∴ $3x(x-1)=0$

∴ $x=0$ and $y=1$, or $x=1$ and $y=0$

The intersections are $(0,1)$ and $(1,0)$.

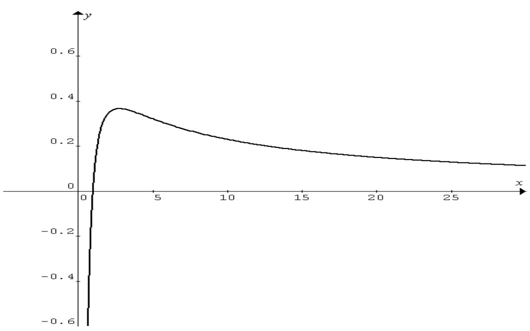
Example 7 $y=\dfrac{x}{x^2+1}$ is the quotient of functions u and v defined by $u(x)=x$ and $v(x)=x^2+1$ respectively.

$D_u=R$ and $D_v=R$, $D_{uv}=D_u\cap D_v=R$



Example 8 $y=\dfrac{\log_e x}{x}$ is the quotient of functions u and v defined by $u(x)=\log_e x$ and $v(x)=x$.

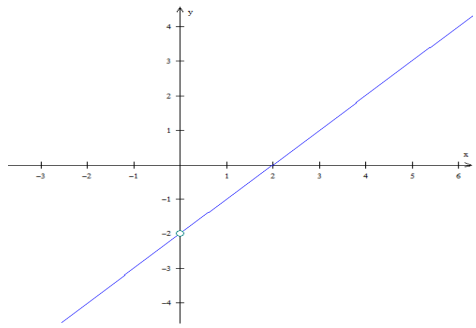
$D_u=R^+$ and $D_v=R\setminus\{0\}$, $D_{uv}=D_u\cap D_v=R^+$



Example 9 Sketch $y=\dfrac{x^2-2x}{x}$ and state its domain and range.

Note: It is wrong to say $y=\dfrac{x^2-2x}{x}=\dfrac{x(x-2)}{x}=x-2$. The correct statement is $y=x-2$ where $x\neq 0$.

In the graph of $y=\dfrac{x^2-2x}{x}$, a small circle is used to indicate the point at $x=0$ is excluded from the sketch.



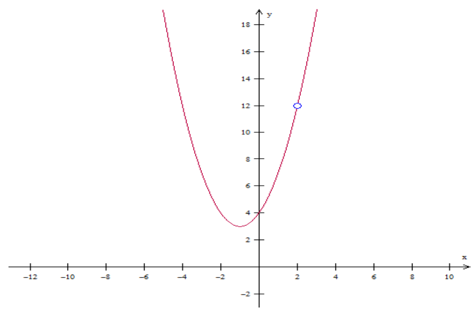
The domain is $R\setminus\{0\}$, the range is $R\setminus\{-2\}$.

Example 10 Sketch $y=\dfrac{x^3-8}{x-2}$ and state its domain and range.

$y=\dfrac{x^3-8}{x-2}$ is equivalent to

$y=\dfrac{x^3-8}{x-2}=\dfrac{(x-2)(x^2+2x+4)}{x-2}=x^2+2x+4$ where $x\neq 2$.

The graph is a parabola but the point at $x=2$ is excluded.



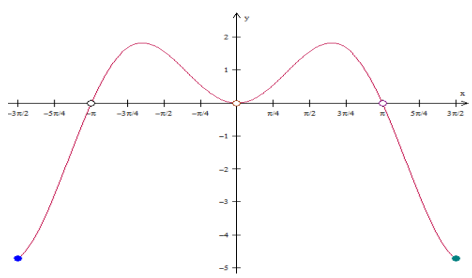
The domain is $R\setminus\{2\}$, the range is $[3,\infty)$.

Example 11 Sketch $y=\dfrac{x\sin^2 x}{\sin x}$ from $x=-\dfrac{3\pi}{2}$ to $x=\dfrac{3\pi}{2}$.

State its domain and range.

$y=\dfrac{x\sin^2 x}{\sin x}$ from $x=-\dfrac{3\pi}{2}$ to $x=\dfrac{3\pi}{2}$ can be stated as

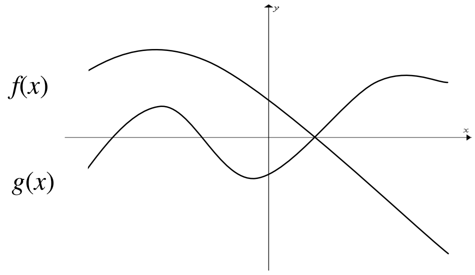
$y=x\sin x$ where $-\dfrac{3\pi}{2}\leq x\leq\dfrac{3\pi}{2}$ and $x\neq -\pi$, 0, π.



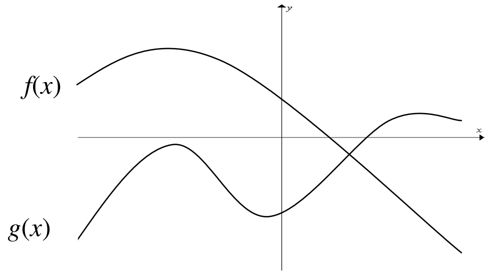
The domain is $\left[-\frac{3\pi}{2},-\pi\right)\cup(-\pi,0)\cup(0,\pi)\cup\left(\pi,\frac{3\pi}{2}\right]$, the range is $\left[-\frac{3\pi}{2},0\right)\cup(0,1.81971]$

Exercise: Next page

Q1 Sketch $f(x)+g(x)$.



Q2 Sketch $f(x)-g(x)$.



Q3 Use addition of ordinates to sketch $y=\frac{x}{2}+\cos x$, $0\le x\le \pi$.

Q4 Use addition of ordinates to sketch $y=x^2-2^x$, $-1\le x\le 1$.

Q5 State the domain and range of $y=x\sqrt{x+1}$ without CAS.

Q6 State the domain of $y=\frac{x^2}{\cos x}$ from $x=-\pi$ to $x=\pi$ without CAS.

Q7 Solve $(x^2-1)(x+2)^7=(x^2-1)(x+2)^6$ for x.

Q8 Solve $(x+1)^2e^x=(x+1)e^x$ for x.

Q9 Sketch $y=\frac{x}{x^2-2x}$ without CAS. State its domain and range.

Q10 Sketch $y=\frac{x-1}{(x-1)^3}$ without CAS. State its domain and range.

Q11 Sketch $y=\frac{\sin^2 x}{x\sin x}$ from $x=-\frac{3\pi}{2}$ to $x=\frac{3\pi}{2}$. State its domain and range.

Numerical, algebraic and worded answers:

5. $[-1,\infty)$, $[0,\infty)$
6. $\left[-\pi,-\frac{\pi}{2}\right)\cup\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
7. $x=-2,\ \pm 1$
8. $x=-1,\ 0$
9. $R\setminus\{0,2\}$, $R\setminus\left\{-\frac{1}{2},0\right\}$
10. $R\setminus\{1\}$, R^+
11. $\left[-\frac{3\pi}{2},-\pi\right)\cup(-\pi,0)\cup(0,\pi)\cup\left(\pi,\frac{3\pi}{2}\right]$, $\left[-\frac{2}{3\pi},0\right)\cup(0,1)$