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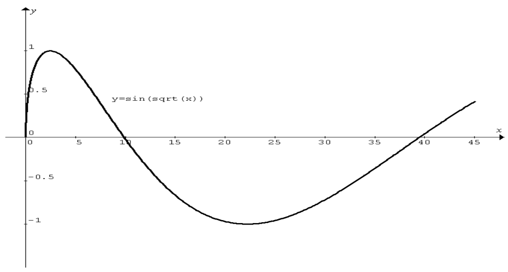
Graphs of composite functions

Given two functions f and g with equations $y = f(x)$ *and* $y = g(x)$ *respectively, new functions can be generated in the following ways:*

In $y = f(x)$ *the variable x is replaced by* $g(x)$, *∴ the equation of the new function is* $y = f(g(x))$.
In $y = g(x)$ *the variable x is replaced by* $f(x)$, *∴ the equation of the new function is* $y = g(f(x))$.
These new functions are called composite functions. They are denoted as $f \circ g$ *and* $g \circ f$ *respectively,*
i.e. $f \circ g(x) = f(g(x))$ *and* $g \circ f(x) = g(f(x))$.

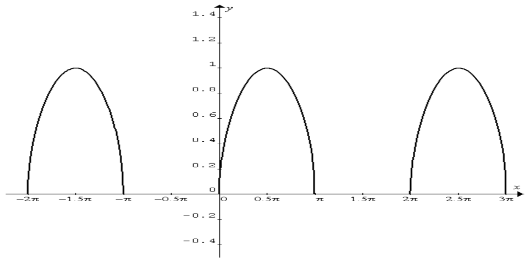
Example 1 Generate two composite functions from functions *f* and *g* defined by $f(x) = \sin x$ and $g(x) = \sqrt{x}$.

Replace *x* by $\sqrt{x}$ in $f(x) = \sin x$ to obtain $f \circ g(x) = \sin(\sqrt{x})$.



$f \circ g(x) = \sin(\sqrt{x})$ is defined for $x \geq 0$.
Hence $D_{f \circ g} = \{x : x \geq 0\}$.

Replace *x* by $\sin x$ in function $g(x) = \sqrt{x}$ to obtain $g \circ f(x) = \sqrt{\sin x}$.



$g \circ f(x) = \sqrt{\sin x}$ is defined for $\sin x \geq 0$,
i.e. $x \in [2n\pi, (2n+1)\pi]$ where $n = 0, \pm 1, \pm 2, ...$
Hence $D_{g \circ f} = \{x : 2n\pi \leq x \leq (2n+1)\pi, n = 0, \pm 1, \pm 2, ...\}$

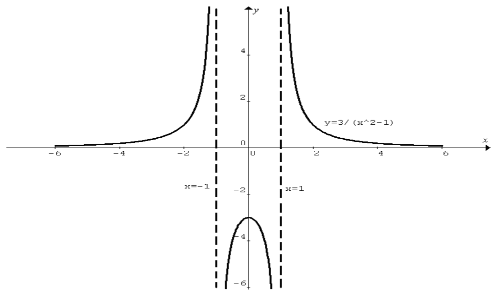
Example 2 Find the domain and range, and sketch the graph of $y = \dfrac{3}{x^2 - 1}$.

$y = \dfrac{3}{x^2 - 1}$ is a composite function, $y = f \circ g(x) = f(g(x))$,

where $f(x) = \dfrac{3}{x}$ and $g(x) = x^2 - 1$.

The function is defined if $x^2 - 1 \neq 0$, i.e. $x \neq \pm 1$.
Hence $D_{f \circ g} = R \setminus \{-1, 1\}$ and the function has vertical asymptotes $x = -1$ and $x = 1$.

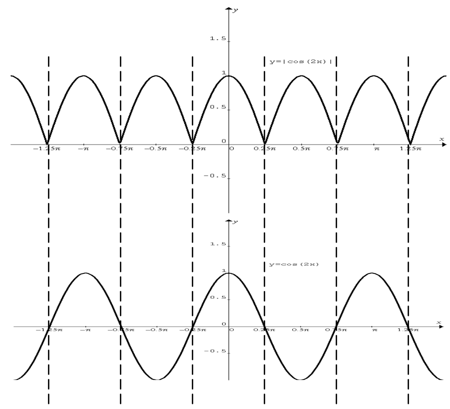
The value of the function cannot be zero, $\therefore R_{f \circ g} = R \setminus \{0\}$ and the function has the *x*-axis as a horizontal asymptote $y = 0$.



Example 3 Find the domain and range, and sketch the graph of $y = |\cos(2x)|$.

$y = |\cos(2x)|$ is a composite function, $y = f \circ g(x) = f(g(x))$, where $f(x) = |x|$ and $g(x) = \cos(2x)$.

$y = |\cos(2x)|$ is defined for $\cos(2x) \in R$, i.e. $x \in R$.
Hence $D_{f \circ g} = R$.
Since $-1 \leq \cos(2x) \leq 1$, $\therefore 0 \leq |\cos(2x)| \leq 1$.
Hence the range of the composite function is $R_{f \circ g} = [0, 1]$.



The graph of $y = \cos(2x)$ is also shown for comparison. The negative half is reflected in the *x*-axis.

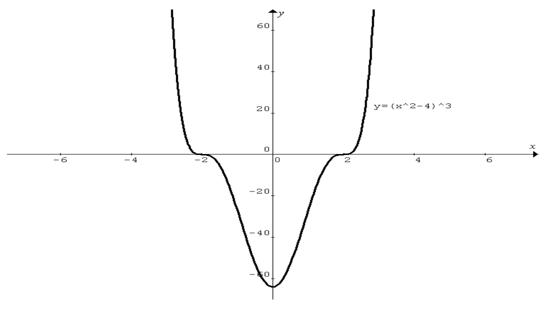
Example 4 Find the domain and range, and sketch the graph of $y = (x^2 - 4)^3$.

$y = (x^2 - 4)^3$ is a composite function $y = f \circ g(x) = f(g(x))$, where $f(x) = x^3$ and $g(x) = x^2 - 4$.

It is defined for all real *x* values. Hence $D_{f \circ g} = R$.

The lowest value of the function is $(-4)^3 = -64$.
Hence $R_{f \circ g} = [-64, \infty)$.

The function can be expressed as the product of the cube of two linear factors, $y = (x^2 - 4)^3 = (x-2)^3(x+2)^3$, $\therefore$ the *x*-intercepts at ± 2 are stationary points of inflection.



Example 5 (VCAA 2011 Exam 1) Given $f(x)=\sqrt{x^2-9}$ and $g(x)=x+5$ (a) express $f(g(x))$ in the form $\sqrt{(x+c)(x+d)}$, (b) state the maximal domain of $f(g(x))$.

(a) $f(g(x))=\sqrt{(g(x))^2-9}=\sqrt{(x+5)^2-9}=\sqrt{(x+2)(x+8)}$

(b) $f(g(x))$ is defined when $(x+2)(x+8)\ge 0$. This is true when $x\le -8$ or $x\ge -2$.

Graphs of inverse relations

*A relation is a set of points. A new set of points can be generated by interchanging the x and y-coordinates of each point. This new set of points is called the **inverse** of the original relation. The equation of the inverse is obtained by interchanging x and y in the original equation.*

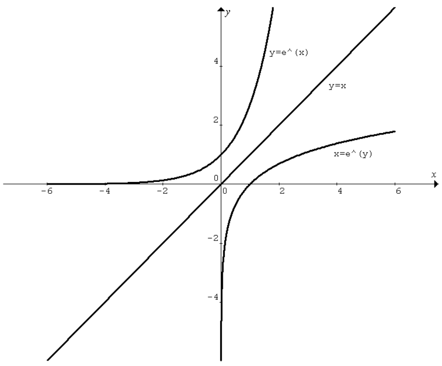
The y-intercepts of the original relation become the x-intercepts of the new relation, and the x-intercepts of the original become the y-intercepts of the new relation.

The horizontal asymptotes of the original relation become the vertical asymptotes of the new relation, and the vertical asymptotes of the original become the horizontal asymptotes of the new relation.

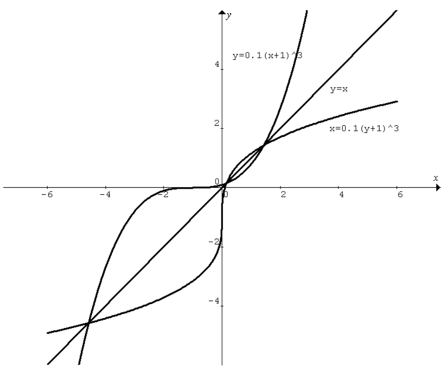
The range of the original relation becomes the domain of the new relation, and the domain of the original becomes the range of the new relation.

Graphically the inverse relation and the original relation are reflections of each other in the line $y=x$.
Note: The same scale for both axes must be used to display this reflection property graphically.

Example 6 The equation of the inverse of $y=e^x$ is $x=e^y$ which can be expressed with *y* as the subject, $y=\log_e x$.



Example 7 The equation of the inverse of $y=0.1(x+1)^3$ is $x=0.1(y+1)^3$ which can be expressed with *y* as the subject, $y=(10x)^{\frac{1}{3}}-1$.

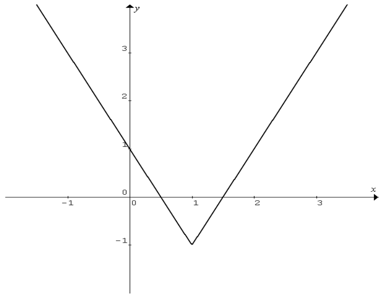


Hybrid functions

*In some practical situations a different rule (equation) is required for a different interval in the domain of the function. Such a function is called a **hybrid function**.*

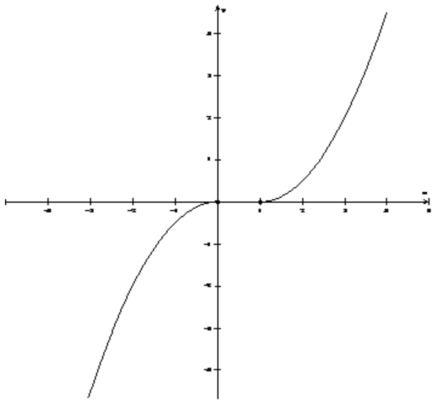
Example 8 The modulus function *f* defined by $f(x)=2|x-1|-1$ can be expressed as a hybrid function

$$f(x)=\begin{cases}2x-3 & x\ge 1\\ 1-2x & x<1\end{cases}$$



Example 9 Sketch the graph of $y=f(x)$ where

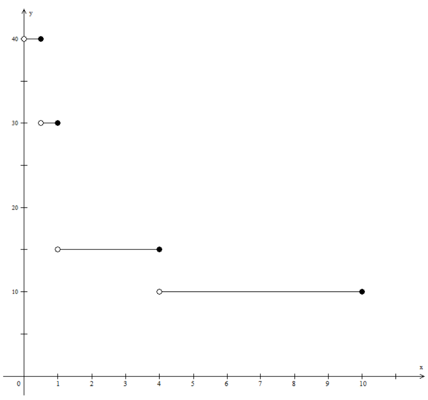
$$f(x)=\begin{cases}\frac{1}{2}(x-1)^2 & x>1\\ 0 & 0<x\le 1\\ -\frac{1}{2}x^2 & x\le 0\end{cases}$$



Example 10 A paint shop only sells paints in 0.5 L, 1 L, 4 L and 10 L cans. The corresponding prices are \$20, \$30, \$60 and \$100. Draw a graph of the cost per L of paint (\$ *y*) as a function of the amount of paint (*x* L) if you buy just *one* can of paint for a job.

Use a hybrid function to display the given information.

$$y=\begin{cases}40 & 0<x\le 0.5\\ 30 & 0.5<x\le 1\\ 15 & 1<x\le 4\\ 10 & 4<x\le 10\end{cases}$$



Exercise: Next page

Q1 Sketch $y = |x^2 - 9| + 3$. State its domain and range. Non- CAS.

Q2 Sketch $y = \left| |x-1|^3 - 1 \right|$. State its domain and range. Non-CAS.

Q3 Given $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x^2}$, state the domain and range of $g(f(x))$. Sketch $y = g(f(x))$. Non- CAS.

Q4 Given $f(x) = \sin(\pi x)$ where $-1 \le x \le 1$, and $g(x) = x^2$ where $x \in R$. Sketch $y = g(f(x))$ and state its domain and range. Non-CAS.

Q5 Find the domain and range, and sketch the graph of $y = (x^2 - 1)^2$. Non-CAS.

Q6 Non-CAS. Sketch $y = \frac{1}{x^2}$ and $x = \frac{1}{y^2}$ on the same set of axes.

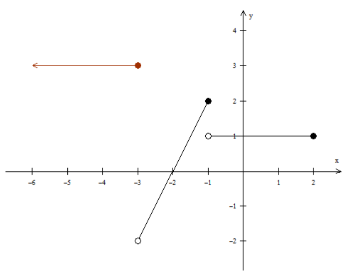
Q7 Non-CAS. Sketch $y = \log_e(x-1)$ and $x = \log_e(y-1)$ on the same set of axes.

Q8 Non-CAS. Sketch $y = \frac{1}{\sqrt{x}}$ and $x = \frac{1}{\sqrt{y}}$ on the same set of axes.

Q9 Express $f(x) = 2|x+1| - 2$ as a hybrid function.

Q10 Non-CAS. Sketch $y = \begin{cases} x^{\frac{1}{3}} + 1 & -1 \le x < 1 \\ 1 - x^{\frac{2}{3}} & 1 \le x < 2 \end{cases}$

Q11 State the hybrid function shown in the graph.



Numerical, algebraic and worded answers:

1. R, $[3, \infty)$ 2. R, $[0, \infty)$

3. $(0, \infty)$, $(0, \infty)$

4. $[-1, 1]$, $[0, 1]$

5. R, $[0, \infty)$

9. $f(x) = \begin{cases} -2x - 4 & x < -1 \\ 2x & x \ge -1 \end{cases}$

11. $y = \begin{cases} 3 & x \le -3 \\ 2x + 4 & -3 < x \le -1 \\ 1 & -1 < x \le 2 \end{cases}$