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Discrete random variables and probability distributions in general

Probability experiment, outcomes and sample space

A probability experiment is a process of data collection that shows variation in its outcomes. The set of all possible distinct outcomes is called the sample space of the experiment.

A random variable X is a numerical valued function defined on a sample space, i.e. if e is an outcome in the sample space, $X(e)$ *is a numerical value assigned to e. The values of X can be* discrete *or* continuous.

Example 1 *Probability experiment*: A fair die is rolled and the uppermost number is recorded.

Many *discrete random variables* can be made up for this experiment. Examples (denoted by capital letters):
P: The number that appears uppermost
Q: Number of odd uppermost numbers
R: Number of uppermost numbers that are less than 3
S: A third of the uppermost number

Probability distributions of P, Q, R and S (each one is displayed as a table showing the possible values of the random variable and the corresponding probabilities):

x	1	2	3	4	5	6
$\Pr(P=x)$	1/6	1/6	1/6	1/6	1/6	1/6

x	0	1
$\Pr(Q=x)$	3/6	3/6

x	0	1
$\Pr(R=x)$	4/6	2/6

x	1/3	2/3	1	4/3	5/3	2
$\Pr(P=x)$	1/6	1/6	1/6	1/6	1/6	1/6

Notes:
(1) $0 \le \Pr \le 1$.
(2) $\sum \Pr = 1$ *for a probability distribution.*
(3) The values of a discrete random variable are not necessarily whole numbers. They can be negative, fractions or decimals.

Example 2 Probability experiment: Two fair dice are rolled and the uppermost numbers are recorded.

Examples of discrete random variables:
A: The difference of the two uppermost numbers
B: Number of even uppermost numbers
C: Number of equal uppermost numbers
D: The average of the two uppermost numbers

Probability distributions:

a	0	1	2	3	4	5
$\Pr(A=a)$	6/36	10/36	8/36	6/36	4/36	2/36

b	0	1	2
$\Pr(B=b)$	9/36	18/36	9/36

c	0	2
$\Pr(C=c)$	30/36	6/36

d	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
$\Pr(D=d)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Example 3 Repeated probability experiment: A fair coin is tossed three times and the result (H or T) of each toss is recorded.

Examples of random variables:
X: Number of tails in the three tosses
Y: Difference between the number of heads and the number of tails in the three tosses
W: Square of the number of heads in the three tosses.

Probability distributions:

x	0	1	2	3
$\Pr(X=x)$	1/8	3/8	3/8	1/8

x	1	3
$\Pr(Y=x)$	6/8	2/8

x	0	1	4	9
$\Pr(W=x)$	1/8	3/8	3/8	1/8

In examples 1, 2 and 3, the outcomes in each experiment are **equally likely**, $\therefore$ the probabilities can be calculated by

$$probability = \frac{number / of\ / favourable / outcomes}{total / number / of\ / outcomes}.$$

Example 4 Random variable V of a probability experiment has the probability distribution shown in the following table.

x	−1.3	1.2	10.3	11.1
$\Pr(V=x)$	0.31	0.22	0.4	0.07

(a) State all the possible values of V.
(b) State the mode of the distribution of V.
(c) Find the probability that $V < 10.3$.
(d) Find the probability that $V < 1.3$.
(e) Find the probability that $0 < V < 15$.

(a) Possible values of V are: −1.3, 1.2, 10.3 and 11.1. There are only four because $\sum \Pr = 0.31 + 0.22 + 0.4 + 0.07 = 1$

(b) **Mode** is the value of a random variable that has the highest probability. In this example, the mode is 10.3.

(c)
$\Pr(V < 10.3) = \Pr(V = -1.3) + \Pr(V = 1.2) = 0.31 + 0.22 = 0.53$

(d) $\Pr(V < 1.3) = \Pr(V = -1.3) + \Pr(V = 1.2) = 0.31 + 0.22 = 0.53$

(e) $\Pr(0 < V < 15) = \Pr(V = 1.2) + \Pr(V = 10.3) + \Pr(V = 11.1)$
$= 0.22 + 0.4 + 0.07 = 0.69$.
Alternatively,
$\Pr(0 < V < 15) = 1 - \Pr(V = -1.3) = 1 - 0.31 = 0.69$.

Example 5 Which one (or more) of the following tables cannot be a probability distribution? Give a reason.

(a)

x	-2	-1	0	1	2	5
$\Pr(X=x)$	0.2	0.3	0.1	0.2	0.1	0.1

(b)

x	2	-1	5	1	-2	0
$\Pr(X=x)$	0.2	0.3	0.1	0.2	0.1	0.1

(c)

x	-2	-1	0	1	2	5
$\Pr(X=x)$	0.2	0.2	0.3	0	0.1	0.2

(d)

x	-2.1	-1.3	0.1	1.7	2	1.6
$\Pr(X=x)$	0.1	0.3	0.1	0.2	0.1	0.2

(e)

x	-2	-1	0	1	-2	5
$\Pr(X=x)$	0.2	0.3	0.1	0.2	0.1	0.1

(f)

x	-2	-1	1/2	1	2	3
$\Pr(X=x)$	0.2	0.3	0.1	0.2	0.2	-0.1

(e) cannot be a probability distribution because it is not a function. (f) cannot be a probability distribution because $\Pr(X=3)<0$.

Example 6 Consider the following probability distribution.

x	-1	-0.5	0	1	1.5	3
$\Pr(X=x)$	0.2	0.3	0.1	0.2	0.1	0.1

Find (a) $\Pr(X\le 2 \mid X\le 0)$, (b) $\Pr(X\le 2\mid X\ge 0)$.

(a) $\Pr(X\le 2\mid X\le 0)=\dfrac{\Pr(X\le 2\cap X\le 0)}{\Pr(X\le 0)}=\dfrac{\Pr(X\le 0)}{\Pr(X\le 0)}=1$

(b) $\Pr(X\le 2\mid X\ge 0)=\dfrac{\Pr(X\ge 0\cap X\le 2)}{\Pr(X\ge 0)}$

$=\dfrac{\Pr(X=0)+\Pr(X=1)+\Pr(X=1.5)}{\Pr(X=0)+\Pr(X=1)+\Pr(X=1.5)+\Pr(X=3)}$

$=\dfrac{0.1+0.2+0.1}{0.1+0.2+0.1+0.1}=0.8$

Example 7 Repeated probability experiment: A coin is biased such that the probability of getting *head* is 0.6 in a single toss. The biased coin is tossed 3 times and the results recorded.
(a) Find the probability distribution of random variable X defined as the number of heads in 3 tosses.
(b) Find $\Pr(X<2)$ and $\Pr(X<2\mid X\le 2)$.

(a)

x	0	1	2	3
$\Pr(X=x)$	0.064	0.288	0.432	0.216

$\Pr(X=0)=0.4\times0.4\times0.4=0.064$

$\Pr(X=1)=3\times0.4\times0.4\times0.6=0.288$

$\Pr(X=2)=3\times0.4\times0.6\times0.6=0.432$

$\Pr(X=3)=0.6\times0.6\times0.6=0.216$

(b) $\Pr(X<2)=\Pr(X=0)+\Pr(X=1)=0.352$

$\Pr(X<2\mid X\le 2)=\dfrac{\Pr(X<2\cap X\le 2)}{\Pr(X\le 2)}=\dfrac{\Pr(X<2)}{\Pr(X\le 2)}$

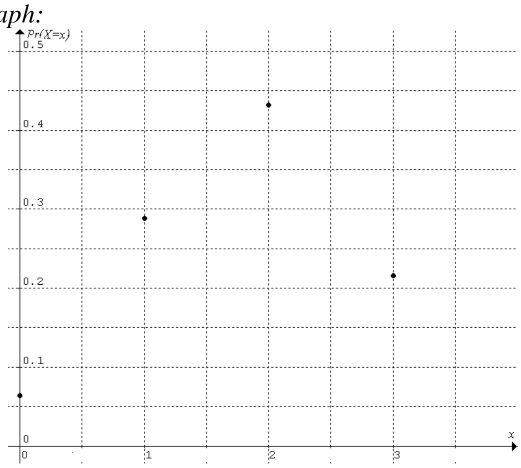
$=\dfrac{0.352}{0.352+0.432}=\dfrac{22}{49}$

Probability distributions as graphs and rules

The probability distribution of a discrete random variable can be displayed as a graph or specified by a rule in some cases instead of a table.

Consider the probability distribution of random variable X in the last example.

As a graph:



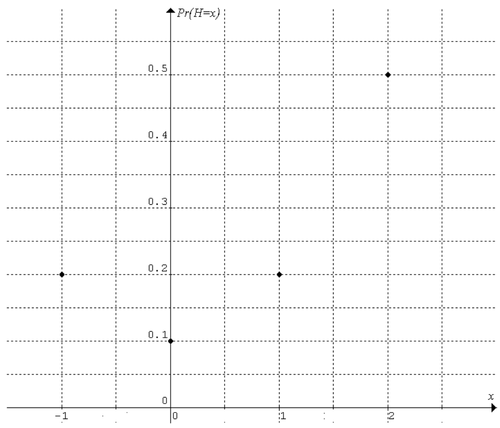
As a rule:

$\Pr(X=x)={}^3C_x(0.6)^x(0.4)^{3-x}$, *where* $x=0, 1, 2, 3$

Example 8 The rule of the probability distribution of random variable H is $\Pr(H=x)=0.1(x^2+1)$, where $x=-1$, 0, 1, 2. Set up a table and plot a graph to display the probability distribution of H.

x	-1	0	1	2
$\Pr(H=x)$	0.2	0.1	0.2	0.5

Note: $\sum \Pr(H=x)=1$



Example 9 The rule of the probability distribution of random variable K is $\Pr(K=x)=\dfrac{e^{1-\frac{x^2}{2}}}{e+2\sqrt{e}}$ for $x=-1$, 0, 1.

Set up a table for the probability distribution of K.

Show that $\sum p(x)=1$ exactly.

x	-1	0	1
$\Pr(K=x)$	~0.274	~0.452	~0.274

$\sum \Pr(K=x)=\dfrac{\sqrt{e}}{e+2\sqrt{e}}+\dfrac{e}{e+2\sqrt{e}}+\dfrac{\sqrt{e}}{e+2\sqrt{e}}=\dfrac{e+2\sqrt{e}}{e+2\sqrt{e}}=1$

Questions: Next page

1. A fair die is rolled and the uppermost number is recorded. Label each of the following as an *outcome*, an *event* or a *random variable* of the probability experiment.

Number of even numbers
Six
Even number

2. Two fair dice are rolled and the uppermost numbers are recorded. Define random variable X as the sum of the uppermost numbers. Set up a table to show the probability distribution of X.

3. Refer to Q2. Find (a) $\Pr(X > 3)$ and (b) $\Pr(X < 10 \mid X > 3)$.

4. Refer to Q3. Explain why the two events $X > 3$ and $X < 10$ are **not** independent.

5. A fair coin is tossed four times and the result (H or T) of each toss is recorded. Define random variable X as the difference between the number of heads and the number of tails. Set up a table to show the probability distribution of X.

6. Random variable X has the probability distribution shown in the following table.

x	-1.5	-1.2	1.3	4.1
$\Pr(X = x)$	0.31	0.22	0.4	0.07

Find (a) $\Pr(X < 1.3)$, (b) $\Pr(X > -1)$ and (c) $\Pr(-2 < X < 2)$.

7. Find the value(s) of k in the following probability distribution of X.

x	1/3	2/3	1	4/3	5/3	2
$\Pr(X = x)$	k^2	$2k^2$	$3k^2$	$4k^2$	$5k^2$	k^2

8. Give two reasons why the following cannot be a probability distribution?

x	3	1.2	0	3	-2	5
$\Pr(X = x)$	0	0.1	0.3	0.2	0.4	0.1

9. The rule of the probability distribution of discrete random variable X is $\Pr(X = x) = ax^2 + 0.5$, where $x = -2$, 1, 4. Find the value of a.

10. Refer to Q9. Display the probability distribution of X graphically.

11. A die is biased with $\Pr(1) = \Pr(2) = \Pr(3) = \frac{2}{9}$ and $\Pr(4) = \Pr(5) = \Pr(6) = \frac{1}{9}$. It is rolled two times and the uppermost numbers are recorded. Tabulate the probability distribution of random variable X defined as the difference between the uppermost numbers.

Numerical, algebraic and worded answers.

1. random variable, outcome, event
2.

x	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X=x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

3. (a) 11/12 (b) 9/11
4. $\Pr(X>3 \cap X<10) \neq \Pr(X>3) \times \Pr(X<10)$
5.

x	0	2	4
$\Pr(X=x)$	3/8	1/2	1/8

6. (a) 0.53 (b) 0.47 (c) 0.93 7. 1/4
8. sum of pr > 1, 2 different probabilities for the same value of x
9. -1/42
11.

x	0	1	2	3	4	5
$\Pr(X=x)$	15/81	24/81	18/81	12/81	8/81	4/81