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Bernoulli sequences and 2-state Markov chains

Here we consider experiments consisting of successive repetitions. Each repetition is called a trial. We assume that there are only two possible results (success S and failure F) for each trial. The use of S and F is to emphasize the point that they are the only possible results and do not bear the same connotation as success or failure in real life, e.g. failing an exam may be labelled as a success. For an experiment consisting of 5 trials, a possible outcome as an example is SFFSF.

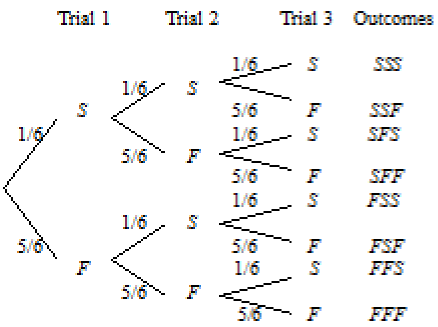
Let p be the probability of success and $q = 1 - p$ *is the probability of failure in a trial.*

If p and q remain constant from trial to trial, then we describe the sequence of trial results as a ***Bernoulli sequence****.*

If p and q depend only on the last trial result (S or F) only, then we describe the sequence of trial results as a ***Markov chain or sequence****.*

The following tree diagrams serve to highlight the difference between the two experiments (consisting of 3 trials each).

Bernoulli sequence: *Roll a fair die 3 times. Success means getting 1, and failure not getting 1.* $p = \frac{1}{6}$ $\therefore q = 1 - \frac{1}{6} = \frac{5}{6}$.



Define random variable X on the sample space as the number of successes in an outcome.

Probability distribution of X:

x	0	1	2	3
$\Pr(X = x)$	$1\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0$	$3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1$	$3\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2$	$1\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3$

Note: $\sum \Pr(X = x) = 1$.

The numbers 1, 3, 3, 1 in the probability calculations are the number of ways in arranging S and F in a sequence, e.g. for the sequence consisting of 2 indistinguishable S's and 1F, there are $\frac{3!}{1!2!} = 3$ *arrangements. They are SSF, SFS and FSS.*

$\frac{3!}{1!2!}$ *is the definition of* ${}^3C_1$ *or* ${}^3C_2$.

In general for a sequence consisting of x S's and $(n - x)$ *F's, the number of arrangements of S and F is* ${}^nC_x$ *or* ${}^nC_{n-x}$.

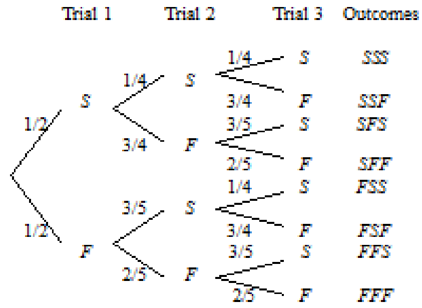
For a Bernoulli sequence consisting of n trials,

$\Pr(X = x) = {}^nC_x\left(p^x\right)(1 - p)^{n-x}$, $x = 0,1,2,3,....,n$.

This is known as the ***binomial distribution*** *of X. The number of trials n and the probability of success p are the parameters of the distribution.*

A binomial distribution with parameters n and p is denoted as $Bi(n, p)$.

Markov chain: *Consider an experiment consisting of 3 trials. Suppose in the first trial, p and q are equal. In the subsequent trials, p and q depend on the preceding trial result.*



Define random variable X on the sample space as the number of successes in an outcome.

$\Pr(X = 0) = \Pr(FFF) = \left(\frac{1}{2}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = 0.08$

$\Pr(X = 1) = \Pr(SFF \cup FSF \cup FFS)$

$= \Pr(SFF) + \Pr(FSF) + \Pr(FFS)$

$= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{2}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{5}\right)\left(\frac{3}{5}\right) = 0.495$

$\Pr(X = 2) = \Pr(SSF \cup SFS \cup FSS)$

$= \Pr(SSF) + \Pr(SFS) + \Pr(FSS)$

$= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{3}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)\left(\frac{1}{4}\right) = 0.39375$

$\Pr(X = 3) = \Pr(SSS) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = 0.03125$

Probability distribution of X:

x	0	1	2	3
$\Pr(X = x)$	0.08	0.495	0.39375	0.03125

Note: $\sum \Pr(X = x) = 1$.

The two distributions above are typical examples of discrete probability distributions.

Example 1 A search party carries 3 emergency signal flairs, each has a probability of lighting of 0.995. Find (a) the probability that at least one flair lights, (b) the probability that exactly two flairs light.

(a) Binomial: $n = 3$, $p = 0.995$, let X be the number of flairs that light. $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - 0.005^3 \approx 1$ or by CAS

(b) $\Pr(X = 2) = {}^3C_2(0.995)^2(0.005) \approx 0.015$ or by CAS

Example 2 A traffic light on the way to school is red 35% of the time. What is the probability of stopping by the red light (a) on 2 consecutive days? (b) on 3 consecutive days? (c) 2 out 3 consecutive days?

(a) Binomial: $n = 2$, $p = 0.35$, let X be the number of days stopped by red light. $\Pr(X = 2) = 0.35 \times 0.35 \approx 0.123$

(b) Binomial: $n = 3$, $p = 0.35$, $\Pr(X = 3) = 0.35^3 \approx 0.043$

(c) Binomial: $n = 3$, $p = 0.35$,

$\Pr(X = 2) = {}^3C_2(0.35)^2(0.65) \approx 0.239$ or by CAS

Example 3 In a science experiment, a student has a 40% chance of getting results. How many times the experiment should be performed if the student wishes to be 95% confident that (a) at least one experiment (b) at least two experiments, will be successful?

Assume the sequence of experiments is a Bernoulli sequence, and has a binomial distribution with $p = 0.4$, and n is to be found.

(a) $\Pr(X \geq 1) \geq 0.95$, $1 - \Pr(X = 0) \geq 0.95$,

$\therefore \Pr(X = 0) \leq 0.05$, ${}^nC_0(0.4^0)(0.6^n) \leq 0.05$,

$\therefore 0.6^n \leq 0.05$, $n\log_e 0.6 \leq \log_e 0.05$, $n \geq \dfrac{\log_e 0.05}{\log_e 0.6} = 5.8645$

$\therefore n \geq 6$.

(b) $\Pr(X \geq 2) \geq 0.95$, $1 - \Pr(X = 0) - \Pr(X = 1) \geq 0.95$,

$\therefore \Pr(X = 0) + \Pr(X = 1) \leq 0.05$,

${}^nC_0(0.4^0)(0.6^n) + {}^nC_1(0.4^1)(0.6^{n-1}) \leq 0.05$,

$\therefore 0.6^n + 0.4n(0.6^{n-1}) \leq 0.05$.

By CAS or graphical method, sketch $y = 0.6^n + 0.4n(0.6^{n-1})$ and $y = 0.05$. Find the intersection at $n = 9.82$.

$\therefore$ for $0.6^n + 0.4n(0.6^{n-1}) \leq 0.05$, $n \geq 10$.

Example 4 Find the least number of tosses of a fair die to ensure that the probability of picking the right number (a) exactly once (b) exactly twice (c) at least twice, is more than 0.3.

(a) n trials, $p = \dfrac{1}{6}$, X number of times the right number is picked. $\Pr(X = 1) \geq 0.3$, ${}^nC_1\left(\dfrac{1}{6}\right)^1\left(\dfrac{5}{6}\right)^{n-1} \geq 0.3$, solve by CAS,

$n \approx 2.268$ $\therefore n = 3$

(b) $\Pr(X = 2) \geq 0.3$, ${}^nC_2\left(\dfrac{1}{6}\right)^2\left(\dfrac{5}{6}\right)^{n-2} \geq 0.3$, solve by CAS,

$n \approx 4.7535$ $\therefore n = 5$

(c) $\Pr(X \geq 2) \geq 0.3$, $\Pr(X \leq 1) \leq 0.7$,

${}^nC_0\left(\dfrac{1}{6}\right)^0\left(\dfrac{5}{6}\right)^n + {}^nC_1\left(\dfrac{1}{6}\right)^1\left(\dfrac{5}{6}\right)^{n-1} \leq 0.7$, solve by CAS,

$n \approx 6.547$ $\therefore n = 7$

Expected value and variance of random variable *X* with a binomial distribution

For a binomial random variable X with parameters n and p,

$E(X) = \sum x\Pr(X = x) = np$ *and*

$Var(X) = E(X^2) - (\mu_X)^2 = np(1-p)$.

Note: These two formulas are for binomial distributions only.

Example 5 In Section A of Methods Exam 2, one mark is allocated to each of the 22 multiple-choice questions, each question having five possible answers. If a student guesses the answer to every question, (a) how many marks will he expect to get for Section A? (b) Find the probability that his mark for Section A is greater than 8.

(a) $Bi(22, 0.2)$, $\mu = np = 22 \times 0.2 = 4.4$

(b) $\Pr(X > 8) \approx 0.02$ by CAS

Example 6 A binomial random variable has a mean of 50 and a standard deviation of 4. Find the parameters n and p of the distribution.

$\mu = np = 50$ …… (1), $\sigma = \sqrt{np(1-p)} = 5$ …… (2)

From (2), $np(1-p) = 25$, $\therefore 50(1-p) = 25$, $1 - p = 0.5$

$\therefore p = 0.5$, $n = 100$

Example 7 From past experience 5% of students of a large school are out of uniform on any school day. Five students are selected randomly from the school to check for uniform. (a) Set up the probability distribution of the number of students out of uniform. (b) Find the probability that at most one is out of uniform. (c) Calculate μ and σ of the distribution.

(a) Probability experiment: Check 5 students for school uniform.
Number of trials $n = 5$ (5 students)
Success: Out of uniform
Probability of success $p = 0.05$ (5%)
Random variable *X*: Number of students out of uniform.
It is a Bernoulli sequence of 5 trials, X~ $Bi(5, 0.05)$

$\Pr(X = 0) = {}^5C_0(0.05)^0(0.95)^5 = 0.774$

$\Pr(X = 1) = {}^5C_1(0.05)^1(0.95)^4 = 0.204$

$\Pr(X = 2) = {}^5C_2(0.05)^2(0.95)^3 = 0.021$

$\Pr(X = 3) = {}^5C_3(0.05)^3(0.95)^2 = 0.001$

$\Pr(X = 4) = {}^5C_4(0.05)^4(0.95)^1 = 0.000$

$\Pr(X = 5) = {}^5C_5(0.05)^5(0.95)^0 = 0.000$ or by CAS.

Binomial distribution of *X*:

x	0	1	2	3	4	5
$\Pr(X = x)$	0.774	0.204	0.021	0.001	0.000	0.000

(b) $\Pr(at.most.one) = \Pr(X \leq 1) = \Pr(X = 0) + \Pr(X = 1)$
$= 0.774 + 0.204 = 0.978$ or by CAS.
The small discrepancy is due to rounding errors.

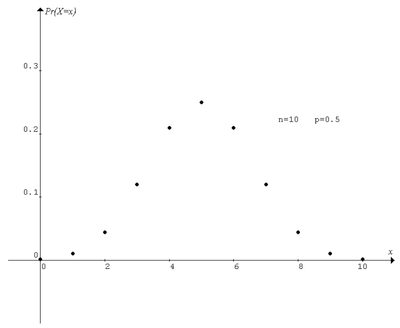
(c) $\mu_X = np = 5 \times 0.05 = 0.25$.

$Var(X) = np(1-p) = 5 \times 0.05 \times 0.95 = 0.2375$,

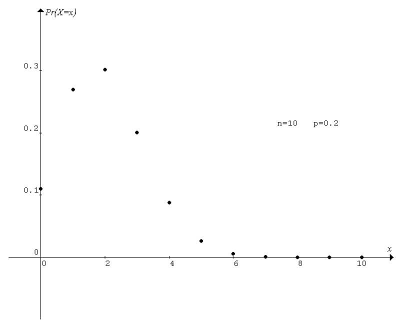
$\therefore \sigma_X = \sqrt{0.2375} = 0.49$.

Effects of *n* and *p* on graphs of binomial distributions

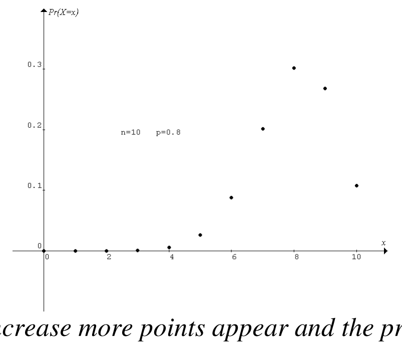
If $p = 0.5$, *the graph of the distribution is bell-shaped and symmetrical about the mean* $\mu = np$.



If $p < 0.5$, *the graph is positively skewed.*



If $p > 0.5$, *the graph is negatively skewed.*



Notes: As n increase more points appear and the probabilities decrease.
If n becomes large, all binomial distributions tend towards bell-shape irrespective of the value of p.

Questions: Next page

1. A soccer player has a probability of 0.6 of scoring a goal with each of 15 attempts. Find the probability of the player scoring more than 7 goals.

2. Refer to Q1. Write a formula for calculating the probability of scoring x goals in 15 attempts.

3. Refer to Q1. Given that the player can score at least 6 goals in 15 attempts, find the probability of the player scoring more than 7 goals.

4. It is known that half of the customers who enter a restaurant order a cup of coffee. Four customers enter the restaurant. Tabulate the probability distribution of random variable X, the number of customers out of the four ordering coffee.

5. Refer to Q4. Determine the mean and the mode of X.

6. A binomial random variable X has a mean of 3 and a standard deviation of $\frac{\sqrt{3}}{2}$. Find $\Pr(X \le 2)$.

7. The following table represents a binomial distribution of X.

x	0	1	2	3
$\Pr(X = x)$	0.614125	0.325125	0.057375	0.003375

Find μ_X and σ_X .

8. Find the minimum number of times that a fair coin can be tossed so that the probability of obtaining a head on each toss is less than 0.0005.

9. Which one or more of the following tables *definitely* cannot be a binomial distribution?

x	1	2	3
$\Pr(X = x)$	a	b	$1-a-b$

x	0	1	2	3
$\Pr(X = x)$	a	b	c	d

x	-1	1	2	3
$\Pr(X = x)$	a	b	c	$1-a-b-c$

10. Every morning there are always two black socks and four white socks in a drawer. Two socks are drawn without replacement in the morning. What is the probability that they are both the same colour?

11. Refer to Q10. What is the probability that different-colour socks are drawn at least two mornings in a week?

Numerical, algebraic and worded answers.

1. 0.79 3. 0.81
2. $\Pr(X = x) = {}^{15}C_x(0.6)^x(0.4)^{15-x}$
4.

x	0	1	2	3	4
$\Pr(X = x)$	0.0625	0.25	0.375	0.25	0.0625

5. mean = 2, mode = 2 6. 0.26
7. mean = 0.45, standard deviation = 0.62
8. 11
9. first and third
10. 7/15 11. 0.96