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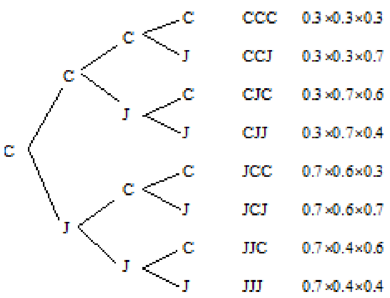
Free maths self-help revision

2-state Markov chains, tree diagrams, transition matrices

Consider a repeated probability experiment. Assume that there are only two possible results (success S and failure F) for each trial. Let p be the probability of success and $q = 1 - p$ *is the probability of failure in a trial. If p and q depend only on the last trial result (S or F) only, then we describe the sequence of trial results as a* ***Markov chain****.*

If all the results of the trials in the chain need to be considered, e.g. find the probability of 3 successes in 5 trials, then a tree diagram is a suitable tool to solve the problem. If the probability of S/F of the final trial is required, then using a transition matrix to solve the problem is the easiest way, especially for large n.

Example 1 A girl either jogs or cycles for her daily exercise. If she jogs one day she cycles the next day with a probability of 0.6. If she cycles one day she jogs the next day with a probability of 0.7. If she cycles today, (a) find the probability that she cycles, jogs and jogs in the next three days.
(b) find the probability that she jogs on exactly two out of the next three days. (c) Tabulate the probability distribution of random variable X – number of days that she jogs in the next three days. (d) Find the mean and mode of X.



(a) $\Pr(CJJ) = 0.3 \times 0.7 \times 0.4 = 0.084$

(b) $\Pr(CJJorJCJorJJC) = 0.084 + 0.7 \times 0.6 \times 0.7 + 0.7 \times 0.4 \times 0.6 = 0.546$

(c)

x	0	1	2	3
$\Pr(X = x)$	0.027	0.315	0.546	0.112

(d) $\mu = 1.743$, mode = 2

Example 2 Refer to example 1. Find the probability that she jogs on day 3 given that she cycles today (take today as day 0).

$\Pr(CCJorCJJorJCJorJJJ) = 0.063 + 0.084 + 0.294 + 0.112 = 0.553$

Transition Matrix and state matrix

By tree diagram it becomes a time-consuming task to find the probability that the girl jogs on, say, day 6 given that she cycles today. A 2×2 transition matrix T consisting of conditional probabilities shortens the calculations. A state matrix S_n *consists of the probabilities of jogging and cycling on day n. Applying the transition matrix on the state matrix gives the state matrix for the following day.*

$S_1 = TS_0$, $S_2 = TS_1$, $S_3 = TS_2$, ……, $S_n = TS_{n-1}$.: $S_n = T^n S_0$

Example 3 Write down the transition matrix and the initial state matrix for example 1.

one day

Transition matrix: next day $\begin{matrix} & C & J \\ C & 0.3 & 0.6 \\ J & 0.7 & 0.4 \end{matrix}$ Initial state matrix: $\begin{matrix} C \\ J \end{matrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Note: The entries in each column in a transition matrix and state matrix always sum to 1. For examples, $\begin{bmatrix} 0.52 & 0.81 \\ 0.48 & 0.19 \end{bmatrix}$ *can be a transition matrix,* $\begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}$ *cannot.*

$\begin{bmatrix} 0.83 \\ 0.19 \end{bmatrix}$ *cannot be a state matrix.*

Example 4 Refer to example 1. Use the transition matrix to find the probability that the girl jogs on day 3 and day 6 given that she cycles on day 0.

$$\begin{matrix} & C \quad J \\ \begin{matrix} C \\ J \end{matrix} & \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}^3 \end{matrix} \underset{day\ 0}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \underset{day\ 3}{\begin{bmatrix} 0.447 \\ 0.553 \end{bmatrix}}$$

.: the probability that she jogs on day 3 is 0.553.

$$\begin{matrix} & C \quad J \\ \begin{matrix} C \\ J \end{matrix} & \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}^6 \end{matrix} \underset{day\ 0}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \underset{day\ 6}{\begin{bmatrix} 0.461931 \\ 0.538069 \end{bmatrix}}$$

.: the probability that she jogs on day 6 is 0.538069.

Note: To avoid confusion always take the initial state as state 0.

Steady state

If a Markov chain continues indefinitely ($n \to \infty$ *) it will eventually converge to a steady state where the probabilities reach their limiting values no matter what the initial state is.*

Example 5 Refer to example 1. Find the probability that the girl jogs on (a) day 10 and (b) day 11. (c) Comment on the answers.
(d) Find S_{10} and S_{11} if $S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (e) Comment on the answers.

(a) $\begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}^{10} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4615416411 \\ 0.5384583589 \end{bmatrix}$

(b) $\begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}^{11} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4615375077 \\ 0.5384624923 \end{bmatrix}$

(c) It appears that each probability approaches a limiting value.

(d) $\begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}^{10} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4615357362 \\ 0.5384642638 \end{bmatrix}$

$\begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}^{11} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4615392791 \\ 0.5384607209 \end{bmatrix}$

(e) Each probability approaches the same limiting value whether $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $S_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Limiting probability values in steady state

If the transition matrix is $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$*, the steady state consisting of limiting probabilities is given by* $\begin{bmatrix} \frac{b}{b+c} \\ \frac{c}{b+c} \end{bmatrix}$ *as* $n \to \infty$.

Example 6 Show the statement above is true.

Let the limiting probabilities be p and q, where $q = 1 - p$

As $n \to \infty$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$, where $a = 1 - c$ and $d = 1 - b$.

.: $ap + bq = p$.: $(1-c)p + b(1-p) = p$

.: $p = \dfrac{b}{b+c}$ and $q = 1 - p = \dfrac{c}{b+c}$.

Example 7 Find the steady state reached under $T=\begin{bmatrix}0.1 & 0.6\\0.9 & 0.4\end{bmatrix}$.

Steady state $S_\infty=\begin{bmatrix}\frac{0.6}{0.6+0.9}\\\frac{0.9}{0.6+0.9}\end{bmatrix}=\begin{bmatrix}0.4\\0.6\end{bmatrix}$.

Note: The steady state matrix can also be found by choosing a large value for n in $\begin{bmatrix}0.1 & 0.6\\0.9 & 0.4\end{bmatrix}^n$, *e.g.* $\begin{bmatrix}0.1 & 0.6\\0.9 & 0.4\end{bmatrix}^{100}=\begin{bmatrix}0.4 & 0.4\\0.6 & 0.6\end{bmatrix}$, *the steady state is* $\begin{bmatrix}0.4\\0.6\end{bmatrix}$.

The two columns in the square matrix have the same values when n is large, indicating the steady state is reached. However, this method does not always give the ***exact*** *limiting probabilities of the steady state as shown in the next example.*

Example 8

Find the **exact** steady state reached under $T=\begin{bmatrix}0.1 & 0.8\\0.9 & 0.2\end{bmatrix}$.

$$\begin{bmatrix}0.1 & 0.8\\0.9 & 0.2\end{bmatrix}^{100}=\begin{bmatrix}0.4705882353 & 0.4705882353\\0.5294117647 & 0.5294117647\end{bmatrix}$$

$$\begin{bmatrix}0.1 & 0.8\\0.9 & 0.2\end{bmatrix}^{200}=\begin{bmatrix}0.4705882353 & 0.4705882353\\0.5294117647 & 0.5294117647\end{bmatrix}$$

$S_\infty=\begin{bmatrix}\frac{0.8}{0.8+0.9}\\\frac{0.9}{0.8+0.9}\end{bmatrix}=\begin{bmatrix}\frac{8}{17}\\\frac{9}{17}\end{bmatrix}$. Note: $0.4705882353\approx\frac{8}{17}$.

Example 9 A worker drinks either tea or coffee only. If he had tea before, the probability that he has coffee in his next cup is 0.43. If he had coffee before, the probability that he has tea next is 0.72. *Suppose that he drinks coffee to start the day.*

(a) Find the probability that he has coffee in his next five cups.
(b) Find the probability that he has tea in his next five cups.
(c) Find the probability that he has at least one cup of tea in his next 5 cups.
(d) Find the probability that he has one cup of tea in his next five cups.
(e) Find the probability that his sixth cup is coffee.
(f) Find the number of cups of drink to reach the steady state, correct to 3 decimal places.

(a) $C\to C\to C\to C\to C\to C$
$(1)(0.28)(0.28)(0.28)(0.28)(0.28)\approx 0.0017$

(b) $C\to T\to T\to T\to T\to T$
$(1)(0.72)(0.57)(0.57)(0.57)(0.57)\approx 0.076$

(c) Pr (at least 1 cup of tea in his next 5 cups) = 1 – Pr (none)
$\approx 1-0.0017=0.9983$

(d) $C\to T\to C\to C\to C\to C$
$(1)(0.72)(0.43)(0.28)(0.28)(0.28)\approx 0.0068$
$C\to C\to T\to C\to C\to C$
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$C\to C\to C\to T\to C\to C$
$(1)(0.28)(0.28)(0.72)(0.43)(0.28)\approx 0.0068$
$C\to C\to C\to C\to T\to C$
$(1)(0.28)(0.28)(0.28)(0.72)(0.43)\approx 0.0068$
$C\to C\to C\to C\to C\to T$
$(1)(0.28)(0.28)(0.28)(0.28)(0.72)\approx 0.0044$
Pr (1 cup of tea in his next 5 cups) $\approx 4\times 0.0068+0.0044=0.032$

(e)

this cup

Transition matrix: next cup $\begin{matrix} & C & T\\ C & 0.28 & 0.43\\ T & 0.72 & 0.57\end{matrix}$ Initial state matrix: $\begin{matrix}C\\T\end{matrix}\begin{bmatrix}1\\0\end{bmatrix}$

Sixth cup state matrix $=\begin{bmatrix}0.28 & 0.43\\0.72 & 0.57\end{bmatrix}^5\begin{bmatrix}1\\0\end{bmatrix}\approx\begin{bmatrix}0.3739\\0.6261\end{bmatrix}$

Probability that his sixth cup is coffee ≈ 0.3739

(f) $\begin{bmatrix}0.28 & 0.43\\0.72 & 0.57\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix}\approx\begin{bmatrix}0.388\\0.612\end{bmatrix}$, $\begin{bmatrix}0.28 & 0.43\\0.72 & 0.57\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix}\approx\begin{bmatrix}0.3718\\0.6282\end{bmatrix}$,

$\begin{bmatrix}0.28 & 0.43\\0.72 & 0.57\end{bmatrix}^4\begin{bmatrix}1\\0\end{bmatrix}\approx\begin{bmatrix}0.37423\\0.62577\end{bmatrix}$, $\begin{bmatrix}0.28 & 0.43\\0.72 & 0.57\end{bmatrix}^5\begin{bmatrix}1\\0\end{bmatrix}\approx\begin{bmatrix}0.37387\\0.62613\end{bmatrix}$,

$\begin{bmatrix}0.28 & 0.43\\0.72 & 0.57\end{bmatrix}^6\begin{bmatrix}1\\0\end{bmatrix}\approx\begin{bmatrix}0.37392\\0.62608\end{bmatrix}$.: 5 cups.

Example 10 (VCAA 2010) Statues in order of completion are sent to an inspector, who classifies them as either ‘Superior’ or ‘Regular’. If a statue is Superior then the probability that the next statue is Superior is p. If a statue is Regular then the probability that the next statue completed is Superior is $p-0.2$. On a particular day, $p=0.9$. On that day
(a) if the first statue inspected is Superior, find the probability that the third statue is Regular
(b) if the first statue inspected is Superior, find the probability that the next three statues are Superior
(c) find the steady state probability that a statue is Superior.
On another day, if the first statue inspected is Superior then the probability that the third statue is Superior is 0.7.
(di) Show that the value of p on this day is 0.75. On this day, a group of 3 consecutive statues is inspected.
(dii) Given that the first statue of the 3 statues is Regular, find the expected number of these 3 statues that will be Superior.

(a) $\Pr(third\_is\_R)=\Pr(SSR)+\Pr(SRR)$
$=1(1-p)p+1(1-p)(1-(p-0.2))=0.12$ when $p=0.9$

(b) $\Pr(SSSS)=1\times p^3=0.9^3=0.729$

(c) $\begin{bmatrix}0.9 & 0.7\\0.1 & 0.3\end{bmatrix}^n\to\begin{bmatrix}0.875 & 0.875\\0.125 & 0.125\end{bmatrix}$ as $n\to\infty$
.: steady state $\Pr(S)=0.875$.

Alternatively, $\Pr(S)=\frac{0.7}{0.7+0.1}=0.875$

(di) The transition matrix is
$\begin{bmatrix}p & p-0.2\\1-p & 1-(p-0.2)\end{bmatrix}=\begin{bmatrix}p & p-0.2\\1-p & 1.2-p\end{bmatrix}$.

State matrix for the first statue is $\begin{bmatrix}1\\0\end{bmatrix}$

State matrix for the second statue is $\begin{bmatrix}p & p-0.2\\1-p & 1.2-p\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}p\\1-p\end{bmatrix}$

State matrix for the third statue is
$\begin{bmatrix}p & p-0.2\\1-p & 1.2-p\end{bmatrix}\begin{bmatrix}p\\1-p\end{bmatrix}=\begin{bmatrix}1.2p-0.2\\\ldots\end{bmatrix}=\begin{bmatrix}0.7\\\ldots\end{bmatrix}$
.: $1.2p-0.2=0.7$, $p=0.75$

(dii) $\Pr(S\mid S)=p=0.75$, $\Pr(R\mid S)=0.25$, $\Pr(S\mid R)=p-0.2=0.55$, $\Pr(R\mid R)=0.45$
$\Pr(no\_S's)=\Pr(RRR)=1\times 0.45\times 0.45=0.2025$
$\Pr(1\_S)=\Pr(RSR)+\Pr(RRS)=1\times 0.55\times 0.25+1\times 0.45\times 0.55=0.385$
$\Pr(2\_S's)=\Pr(RSS)=1\times 0.55\times 0.75=0.4125$
A tree diagram helps to compute the probabilities.

x	0	1	2
$\Pr(X=x)$	0.2025	0.385	0.4125

$E(x)=0\times 0.2025+1\times 0.385+2\times 0.4125=1.21$

Questions: Next page

Suppose the probability that you are late to school if you are late the day before is 0.15, and the probability that you are on time if you are on time the day before is 0.90.

1. Find the transition matrix that can be used to represent the information above.

2. Find the probability that you are on time on Friday if you are on time the Monday before.

3. If you are on time on Monday, find the probability that you are on time the next four days.

4. If you are on time on Monday, find the probability that you are late to school the next four days.

5. If the probability that you are late on Tuesday and Wednesday, and on time on Thursday and Friday, is 0.01 approximately, tell us whether you are late or on time the Monday before.

6. Find the probability that you are late only once in a week if you are on time on Monday.

7. Find the probability that you are late at least once in a week if you are on time on Monday.

8. Given that you are late to school on Tuesday, draw a tree diagram to show the Markov chain accompanied with conditional probabilities for the next 3 days.

9. Tabulate the probability distribution of the number of times that you are late to school for the 3 days in Q8.

10. Calculate the expectation of the number of times that you are late to school for the 3 days in Q8.

11. Find the fraction $\dfrac{\text{number of days late to school}}{\text{number of days attending school}}$ in the long run.

Numerical, algebraic and worded answers.

1. $\begin{bmatrix} 0.15 & 0.10 \\ 0.85 & 0.90 \end{bmatrix}$ 2. 0.8947 3. 0.6561 4. 0.0003375 5. on time

6. 0.27945 7. 0.3439 10. 0.36 approx 11. 2/19

9.

x	0	1	2	3
$\Pr(X = x)$	0.6885	0.2635	0.044625	0.003375